What Your Colleagues Are Saying . . .

"*Mathematize It!* is a must have for every middle school teacher! The authors provide clear guidance and suggestions to help our students become effective problem solvers. The opportunities provided to ponder and practice, as well as the student sample work and videos, make this a resource you will grow professionally from and positively impact student learning. I wish I had this resource when I started teaching 25 years ago!"

Kevin Dykema
8th Grade Math Teacher, Mattawan (MI) Middle School
Author, *Productive Math Struggle*
NCTM Board of Directors, 2016–2019

"This dynamic author trio brings years of classroom experiences to one of the central problems of teaching and learning mathematics: making sense of word problems. Focusing on the construct of 'mathematizing'—drawing, constructing, describing, representing, and making sense of situations—this clear and practical guide needs to be required reading and discussion fodder for every middle school teacher of mathematics. It's just that clear, classroom grounded, and insightful!"

Steve Leinwand
Principal Researcher
American Institutes for Research, Washington, DC

"When I read *Mathematize It!*, I experienced a journey of extending the way I think about mathematical ideas. Readers can develop their mathematical knowledge for teaching by reading this book. Also, with their sandbox approach, the authors describe a problem-solving process that is playful and exploratory. The sandbox process also orients problem solvers toward seeing mathematical structures and relationships. I recommend reading this book to experience deep mathematics and, in turn, learn more about how to support students with engaging in mathematizing."

Amanda Jensen
Professor of Mathematics Education
University of Delaware

"The list of generational math books to come along and truly synthesize what we know so far and what we need to know is a very short and exclusive list. Well, you can confidently add *Mathematize It!* to this collection. Written by three of the most respected math educators today, the book zeros in on that often poorly traveled journey between the question and answer in problem solving. *Mathematize It!* will be your go-to resource to install the mathematical play revolution in elementary classes everywhere!"

Sunil Singh
Author of *Pi of Life: The Hidden Happiness of Mathematics*
and *Math Recess: Playful Learning in an Age of Disruption*

"*Mathematize It!* is a must-read for anyone who has struggled to teach word problems and is ready to figure out what *really* works. The authors present a plethora of strategies that help students focus on the *thinking* part of the problem-solving process while gently helping the reader understand that so many of our 'tried-and-true' methods, such as key words, really don't work. They help us realize that the real work of solving word problems is in the sense-making phase—once students have made sense of a problem, calculating the solution is the simpler part of the process."

Kimberly Rimbey
National Board Certified Teacher
Co-Founder & CEO, KP Mathematics

"*Mathematize It!* addresses the complexity of problem solving more completely than any other individual resource. It is easy to say that we must teach students to 'mathematize situations' but this book helps us to actually help students learn to do it. The challenge and reflection pieces at the end of each chapter are a game changer for unveiling teaching opportunities, prompting discussion in your PLC, and moving this from a book on the professional shelf to a powerful tool to impact instruction."

Gina Kilday
Math Interventionist and MTSS Coordinator
Metcalf Elementary School, Exeter, RI
Presidential Award for Excellence in Mathematics and Science Teaching Awardee
Former Member of the NCTM Board of Directors

"*Mathematize It!* is a book that should be on the shelf of every classroom teacher and division leader who supports mathematics teaching and leading. This valuable resource helps educators to think about the what, why, and the how to make sense of word problems. It gives a framework and visuals on how to support teachers' understanding around problem types and solving problems and excels in assisting teachers in how to make a commitment to teaching for greater understanding."

Spencer Jamieson
Past President, Virginia Council for Mathematics Supervision (VCMS)
Mathematics Specialist for Virginia Council of Teachers of Mathematics (VCTM)

"This is a game changer … even after 20 years of supporting students and their sensemaking of word problems, I am thrilled to learn even more from this trio of authors. They offer practical suggestions, opportunities for practice, and relevant research in order to increase awareness of best practices surrounding word problems. The only key word in this case is MATHEMATIZE! To have this resource in your hands is to have an invitation to the 'mathematizing sandbox'."

Beth Terry
Mathematics Coach
2004 Presidential Award for Excellence in Mathematics and Science Teaching Awardee
Riffa Views International School, Bahrain

"As our students begin to mathematize the world around them, it becomes extremely important that we listen to their thinking so that we can continue to move their understanding forward. What makes *Mathematize It!* such a useful tool for teachers is that it thoughtfully unpacks student strategies, which helps inform and guide our next move as a classroom teacher."

Graham Fletcher
Math Specialist
Atlanta, GA

"*Mathematize It!* engages readers deeply in the mathematics content through an easy-to-use visual analogy: playing in a sandbox. The authors have found a way to make problem-solving seem like a fun task—one that is akin to something we've all been doing forever: playing. Their clever and applicable problem-solving model of thinking provides a structure teachers can use to support students in tackling word problems and actually enjoying the process. It's time for you to play in the sandbox and more importantly, *Mathematize It!*"

Hilary Kreisberg
Director, Center for Mathematics Achievement
Lesley University, Cambridge, MA
Author of *Adding Parents to the Equation*

"The authors provide a detailed and practical guide on how to take a word problem, uncover the mathematics embedded in it, carefully consider representations, and use it all to solve the problem. The reader begins to realize that all models are not created equal. The authors' careful attention to the nuances within mathematical relationships illustrates how mathematizing differs from answer getting, yet shows us that ideas like operation sense and computation are related. The authors' plain-language explanations empower us to leverage those relationships in order to help students become better mathematicians."

Paul Gray
Chief Curriculum Officer, Cosenza & Associates, LLC
Past President, Texas Council of Teachers of Mathematics
NCTM Representative for NCSM: Leadership in Mathematics Education

"I can't wait to use *Mathematize It!* in my work with teachers and students! The excellent examples, including actual student work and teacher commentaries, highlight the complexity of the problem situations in a way that is clear and usable for classroom teachers and for those of us who support them. The focus on operation sense, understanding the role that each quantity plays, and connecting representations to problems makes this a must read for anyone helping students become successful problem solvers. I especially appreciate the inclusion of non-whole-number examples!"

Julie McNamara
Associate Professor
Author of *Beyond Pizzas & Pies* (With Meghan Shaughnessy) and *Beyond Invert & Multiply*
California State University, East Bay, Hayward, CA

"This book is a must-have for anyone who has faced the challenge of teaching problem solving. The ideas to be learned are supported with a noticeably rich collection of classroom-ready problems, examples of student thinking, and videos. Problem solving is at the center of learning and doing mathematics. And so, *Mathematize It!* should be at the center of every teacher's collection of instructional resources."

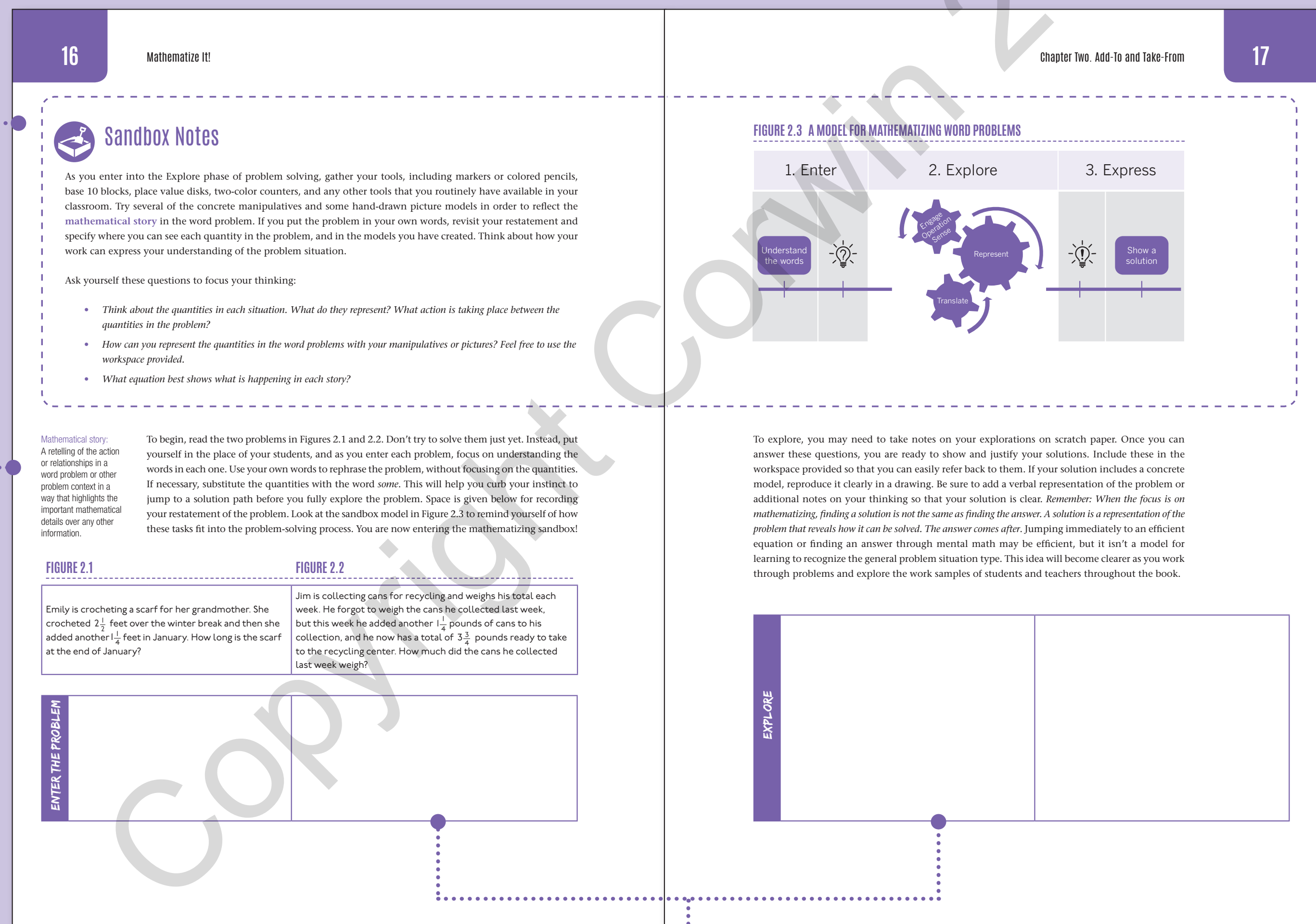
John SanGiovanni
Coordinator, Elementary Mathematics
Howard County Public School System, Ellicott City, MD

"Finally! An answer for equipping students in making sense of word problems. *Mathematize It!* clarifies the challenges in problem solving and gives concrete steps and advice on understanding problem contexts and the mathematics involved. The examples, student work, and videos throughout the book bring ideas to life, and make their implementation doable. This is a must-read for every math teacher who desires their students to truly understand the role of mathematics in the world."

Nanci N. Smith
Associate Professor, Mathematics and Education
Arizona Christian University, Glendale, AZ
Author of *Every Math Learner*

Mathematize It!
The Book at a Glance

Every chapter allows you to play and practice in the **mathematizing sandbox** and do some problem solving yourself!



Sandbox Notes

As you enter into the Explore phase of problem solving, gather your tools, including markers or colored pencils, base 10 blocks, place value disks, two-color counters, and any other tools that you routinely have available in your classroom. Try several of the concrete manipulatives and some hand-drawn picture models in order to reflect the **mathematical story** in the word problem. If you put the problem in your own words, revisit your restatement and specify where you can see each quantity in the problem, and in the models you have created. Think about how your work can express your understanding of the problem situation.

Ask yourself these questions to focus your thinking:

- *Think about the quantities in each situation. What do they represent? What action is taking place between the quantities in the problem?*
- *How can you represent the quantities in the word problems with your manipulatives or pictures? Feel free to use the workspace provided.*
- *What equation best shows what is happening in each story?*

Mathematical story: A retelling of the action or relationships in a word problem or other problem context in a way that highlights the important mathematical details over any other information.

To begin, read the two problems in Figures 2.1 and 2.2. Don't try to solve them just yet. Instead, put yourself in the place of your students, and as you enter each problem, focus on understanding the words in each one. Use your own words to rephrase the problem, without focusing on the quantities. If necessary, substitute the quantities with the word *some*. This will help you curb your instinct to jump to a solution path before you fully explore the problem. Space is given below for recording your restatement of the problem. Look at the sandbox model in Figure 2.3 to remind yourself of how these tasks fit into the problem-solving process. You are now entering the mathematizing sandbox!

FIGURE 2.1	FIGURE 2.2
Emily is crocheting a scarf for her grandmother. She crocheted $2\frac{1}{2}$ feet over the winter break and then she added another $1\frac{1}{4}$ feet in January. How long is the scarf at the end of January?	Jim is collecting cans for recycling and weighs his total each week. He forgot to weigh the cans he collected last week, but this week he added another $1\frac{1}{4}$ pounds of cans to his collection, and he now has a total of $3\frac{3}{4}$ pounds ready to take to the recycling center. How much did the cans he collected last week weigh?

ENTER THE PROBLEM

FIGURE 2.3 A MODEL FOR MATHEMATIZING WORD PROBLEMS

To explore, you may need to take notes on your explorations on scratch paper. Once you can answer these questions, you are ready to show and justify your solutions. Include these in the workspace provided so that you can easily refer back to them. If your solution includes a concrete model, reproduce it clearly in a drawing. Be sure to add a verbal representation of the problem or additional notes on your thinking so that your solution is clear. *Remember: When the focus is on mathematizing, finding a solution is not the same as finding the answer. A solution is a representation of the problem that reveals how it can be solved. The answer comes after.* Jumping immediately to an efficient equation or finding an answer through mental math may be efficient, but it isn't a model for learning to recognize the general problem situation type. This idea will become clearer as you work through problems and explore the work samples of students and teachers throughout the book.

EXPLORE

Marginal definitions throughout for easy reference.

Practice the same problem-solving process your students will in these spaces as you **enter** and **explore** the word problem.

Explore how students use various representations to mathematize and describe their problem-solving process.

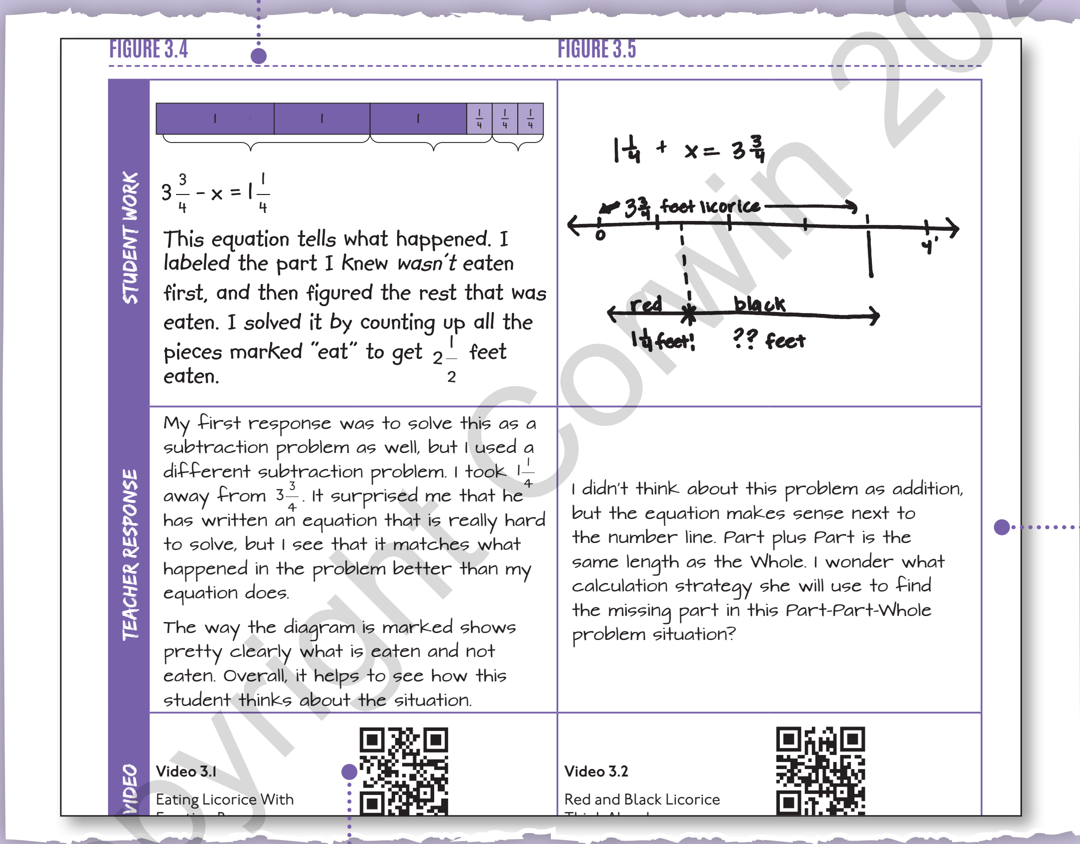


	FIGURE 3.4	FIGURE 3.5
STUDENT WORK	$3\frac{3}{4} - x = 1\frac{1}{4}$ This equation tells what happened. I labeled the part I knew *wasn't* eaten first, and then figured the rest that was eaten. I solved it by counting up all the pieces marked "eat" to get $2\frac{1}{2}$ feet eaten.	$1\frac{1}{4} + x = 3\frac{3}{4}$
TEACHER RESPONSE	My first response was to solve this as a subtraction problem as well, but I used a different subtraction problem. I took $1\frac{1}{4}$ away from $3\frac{3}{4}$. It surprised me that he has written an equation that is really hard to solve, but I see that it matches what happened in the problem better than my equation does. The way the diagram is marked shows pretty clearly what is eaten and not eaten. Overall, it helps to see how this student thinks about the situation.	I didn't think about this problem as addition, but the equation makes sense next to the number line. Part plus Part is the same length as the Whole. I wonder what calculation strategy she will use to find the missing part in this Part-Part-Whole problem situation?
VIDEO	Video 3.1 Eating Licorice With	Video 3.2 Red and Black Licorice

QR codes link to **videos** that actively demonstrate problem-solving thinking with manipulatives and drawings.

Learn from teachers' reflections on student work.

Easy-reference charts launch each chapter to help you make sense of and navigate different problem types.

Addition and Subtraction Problems Situations

	Total Unknown	One Part Unknown	Both Parts Unknown
Part-Part-Whole	The local ice cream shop asked customers to vote for their favorite new flavor of ice cream. 119 customers preferred mint chocolate chip ice cream. 37 preferred açai berry ice cream. How many customers voted? $119 + 37 = x$ $x - 119 = 37$	The local ice cream shop asked customers which new ice cream flavor they like best. 156 customers voted. 37 customers preferred açai berry ice cream. The rest voted for mint chocolate chip ice cream. How many customers voted for mint chocolate chip ice cream? $37 + x = 156$ $x = 156 - 37$	The local ice cream shop held a vote for their favorite new flavor of ice cream. The options were mint chocolate chip and açai berry ice cream. What are some possible combinations of votes? $x + y = 156$ $156 - x = y$

ACTIVE SITUATIONS

	Result Unknown	Change Addend Unknown	Start Addend Unknown	
Add-To	Paulo paid \$4.53 for his sandwich. Then he added \$1.50 for a carton of milk to finish his lunch. How much was his lunch? $4.53 + 1.5 = x$ $4.53 = x - 1.5$	Paulo paid \$4.53 for the sandwich in his lunch. Then he added a carton of milk to his tray to finish his lunch. The total for his lunch is \$6.03. How much is a carton of milk? $4.53 + x = 6.03$ $4.53 = 6.03 - x$	Paulo added a sandwich to his tray. He added a carton of milk that cost \$1.50 to his tray. With the sandwich and milk, his lunch cost \$6.03. How much does the sandwich cost? $x + 1.5 = 6.03$ $6.03 - 1.5 = x$	
Take-From	There are 186 students in the 7th grade. 35 left to get ready to play in the band at the assembly. How many students are not in the band? $186 - 35 = x$ $35 + x = 186$	There are 186 students in the 7th grade. After the band students left class for the assembly, there were 151 students still in their classrooms. How many students are in the band? $186 - x = 151$ $x + 151 = 186$	35 band students left class to get ready to play in the assembly. There were 151 students left in the classrooms. How many students are in the 7th grade? $x - 35 = 151$ $35 + 151 = x$	

RELATIONSHIP (NONACTIVE) SITUATIONS

	Total Unknown	One Part Unknown		Both Parts Unknown
Part-Part-Whole	The local ice cream shop asked customers to vote for their favorite new flavor of ice cream. 119 customers preferred mint chocolate chip ice cream. 37 preferred açai berry ice cream. How many customers voted? $119 + 37 = x$ $x - 119 = 37$	The local ice cream shop asked customers which new ice cream flavor they like best. 156 customers voted. 37 customers preferred açai berry ice cream. The rest voted for mint chocolate chip ice cream. How many customers voted for mint chocolate chip ice cream? $37 + x = 156$ $x = 156 - 37$		The local ice cream shop held a vote for their favorite new flavor of ice cream. The options were mint chocolate chip and açai berry ice cream. What are some possible combinations of votes? $x + y = 156$ $156 - x = y$
	Difference Unknown	**Greater Quantity Unknown**	**Lesser Quantity Unknown**	
Additive Comparison	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 71 cards. How many fewer cards does Roberto have than Jessie? $53 + x = 71$ $53 = 71 - x$	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 18 more cards than Roberto. How many baseball cards does Jessie have? $53 + 18 = x$ $x - 18 = 53$	Jessie and Roberto both collect baseball cards. Jessie has 71 cards and Roberto has 18 fewer cards than Jessie. How many baseball cards does Roberto have? $71 - 18 = x$ $x + 18 = 71$	

Note: These representations for the problem situations in this table reflect our understanding based on a number of resources. These include the tables in the Common Core State Standards for mathematics (CCSS-M; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010); the problem situations as described in the Cognitively Guided Instruction research (Carpenter, Hiebert, & Moser, 1981), in Heller and Greeno (1979), and in Riley, Greeno, and Heller (1984); and other tools. See the Appendix and the companion website for a more detailed summary of the documents that informed our development of these tables.

Key Ideas for quick-referencing and recaps.

KEY IDEAS

1. Addition and subtraction problems that reflect action are called Add-To and Take-From problem situations.
2. They follow a common structure in which there is a starting value (the beginning of the story), a change, and a result (the end of the story). Any one of the three parts of the problem may be the missing value.
3. The variations of Add-To and Take-From have a developmental sequence for proficiency in the Common Core State Standards learning progressions (McCallum et al., n.d.). Result Unknown problems are the most accessible, followed by Change Unknown. The Start Unknown variation is typically the most challenging for students.
4. Students should have experience with all problem situations at all grades, using the complexity of numbers determined by state standards for each grade.
5. An action problem situation can and should be retold in the student's own words. This "mathematical story" simplifies the details of the problem and focuses on the action.
6. When focusing on the meaning of word problems, students should focus on creating concrete or visual (often pictorial), verbal, and symbolic representations that closely match the action (or story) in the problem situation.
7. Students' understanding is also displayed through verbal and symbolic representations. To gauge, clarify, and extend student thinking, ask students to share the story in written or spoken form. Ask questions such as the following:
   - Tell me what happened in your own words.
   - Try telling me what happened without using any numbers.
   - Show me how you're representing the quantities in this situation with your manipulatives (or pictures).
   - Is there action happening in this story? Where?
   - What equation best shows what is happening in this mathematical story?
8. There are four conceptual models for integer addition and subtraction situations:
   a *Translation* models emphasize magnitude and direction.
   b *Relativity* models measure distance from a point that may not be zero.
   c *Bookkeeping* models show gains and losses and borrowing and loaning contexts.
   d *Counterbalance* models preserve all positive and negative values and balance each other.
   e A fifth category of models involves answers found with rules; these are not based on a conceptual model.

Try It Out! to implement, practice, and review the key ideas of the chapter. Practice with a partner or group!

TRY IT OUT!

IDENTIFY THE PROBLEM SITUATION

Classify each of the following problems as an Add-To or Take-From situation. Which term is missing in each problem? Write an equation that matches the problem situation. With a team of colleagues discuss how you decided on the symbolic notation you chose. For example, discuss whether the context should "subtract a positive" or "add a negative" and why.

1. On a day in July, the high temperature in Palm Springs, California, was 103°F at 2 p.m. This is an increase of 20 degrees from the low temperature measured at 4 a.m. What was the temperature at 4 a.m.?
2. Greg launched his drone in a valley located 20′ below sea level. The drone flew straight up and the sensors said that it was 250′ above sea level. How much altitude did the drone gain from its initial launch position?
3. The bakery spent $203.50 on ingredients for blueberry muffins. They sold all the muffins but lost $45 after they repaid the cost of the ingredients. How much money did they take in when they sold the muffins?
4. Paulette was filling her car at the gas station. Before she filled the tank, the gas gauge read $\frac{3}{8}$ of a tank. After she filled the tank, the gauge read $\frac{7}{8}$ of a tank. How much gas did Paulette add to the gas tank?

Reflect sections give you an opportunity to bring some of your knowledge and resources to implementation. Share your answers with a partner or group.

REFLECT

1. We are using the "mathematical story" analogy to help students understand the elements of a word problem. What other tools for understanding narratives do students use in their English/Language Arts curriculum that you can use to support this thinking? You can read the common ELA standards at http://www.corestandards.org/ELA-Literacy/, but this is also a good opportunity to make cross-curricular connections in a meaningful way with colleagues.
2. Number lines are commonly used to represent integer computation. The submarine problem in Figure 2.15 lends itself to a number line representation as a natural extension of the student's diagram in Figure 2.17. How might you use a worked example like the one shown in Figure 2.17 to help students understand how to represent this problem (and others like it) on a number line?
3. Look at the word problems in the textbook you currently use and identify 8–10 Add-To and Take-From problems. Sort them based on which term is missing: start, change, or result. Rewrite the problems to move the missing term. For example, Result Unknown problems could be rewritten as Start Unknown or Change Unknown problems.

Opportunities to stop and think about and create new problems within each problem type.

FIGURE 4.6 ADDITIVE COMPARISON SITUATIONS

CONTEXT	LESSER QUANTITY	DIFFERENCE	GREATER QUANTITY
Weight of pets	17 pounds	5 pounds	x
Baking temperature	325°F	x	350°F
Snowfall	x	3.1 inches	12.6 inches
Money saved			
Pounds of recycling collected			
Length of inchworm			

Mathematatize It!

Grades 6–8

Mathematize It!

Going Beyond Key Words to Make Sense of Word Problems

Grades 6–8

Kimberly Morrow-Leong,
Sara Delano Moore, and
Linda M. Gojak

CORWIN Mathematics

CORWIN
A SAGE Publishing Company

For information:

Corwin
A SAGE Company
2455 Teller Road
Thousand Oaks, California 91320
(800) 233–9936
www.corwin.com

SAGE Publications Ltd.
1 Oliver's Yard
55 City Road
London, EC1Y 1SP
United Kingdom

SAGE Publications India Pvt. Ltd.
B 1/I 1 Mohan Cooperative
Industrial Area
Mathura Road, New Delhi 110 044
India

SAGE Publications Asia-Pacific Pte. Ltd.
18 Cross Street #10–10/11/12
China Square Central
Singapore 048423

Publisher, Mathematics: Erin Null
Associate Content Development Editor: Jessica Vidal
Editorial Assistant: Caroline Timmings
Production Editor: Tori Mirsadjadi
Copy Editor: Amy Marks
Typesetter: Integra
Proofreader: Susan Schon
Indexer: Integra
Cover and Interior Designer: Scott Van Atta
Marketing Manager: Margaret O'Connor

Printed in the United States of America.

Library of Congress Cataloging-in-Publication Data

Names: Morrow-Leong, Kimberly, author. | Moore, Sara Delano, 1966- author. | Gojak, Linda, author.
Title: Mathematize it! : going beyond key words to make sense of word problems, grades 6-8 / Kimberly Morrow Leong, Sara Delano Moore, and Linda M. Gojak.
Description: Thousand Oaks, California : Corwin, [2021] | Includes bibliographical references and index.
Identifiers: LCCN 2020012020 | ISBN 9781506354484 (paperback) | ISBN 9781071819876 (epub) | ISBN 9781071819869 (ebook) | ISBN 9781071819852 (adobe pdf)
Subjects: LCSH: Word problems (Mathematics) | Mathematics—Study and teaching (Middle school)
Classification: LCC QA63 .M656 2021 | DDC 510.71/2—dc23 LC record available at https://lccn.loc.gov/2020012020

This book is printed on acid-free paper.

20 21 22 23 24 10 9 8 7 6 5 4 3 2 1

Contents

List of Videos xix

Preface xxi

Acknowledgments xxiv

About the Authors xxvii

CHAPTER ONE. INTRODUCTION: WHY YOU NEED TO TEACH STUDENTS TO MATHEMATIZE 1

- Problem-Solving Strategies Gone Wrong 2
- What Is Mathematizing? Why Is It Important? 4
  - Focusing on Operation Sense 4
  - Using Mathematical Representations 5
- Teaching Students to Mathematize 7
  - Building Your Understanding of the Operations and Related Problem Situations 7
  - Exploring in the Mathematizing Sandbox: A Problem-Solving Model 10
- A Note About Negative Values 12
- Final Words Before You Dive In 13

CHAPTER TWO. ADD-TO AND TAKE-FROM 14

- Thinking About Active Addition and Subtraction 14
  - Students and Teachers Think About the Problems 18
  - Finding the Unknown, Three Story Structures 18
  - Story Structures: Implications for Teaching 20
  - Focus on Take-From Situations 23
  - Complicating Things: The Start Unknown Variation 24
- Teaching Students to Use Concrete and Pictorial Models 27
- Extending to Negative Values 28
  - Students and Teachers Think About the Problems 29
  - Reasoning and Notating With Negative Values 33
  - Conceptual Models of Integer Contexts 34
  - The Importance of Clear Notation 37
- Key Ideas 40
- Try It Out! 41
- Reflect 43

CHAPTER THREE. PART-PART-WHOLE 44

Thinking About Part-Part-Whole Situations 44
- Students and Teachers Think About the Problems 48
- Defining the Part-Part-Whole Situation 49
- Modeling Relationships Versus Action 49
- Writing Equations: Addition or Subtraction 50

Extending to Negative Values 53
- Students and Teachers Think About the Problems 54
- Representing Part-Part-Whole With Two-Color Counters 55
- Blurred Lines Between Part-Part-Whole and Take-From 58
- The Special Case of Both Parts Unknown 60

Key Ideas 62

Try It Out! 63

Reflect 64

CHAPTER FOUR. ADDITIVE COMPARISON: ANOTHER KIND OF RELATIONSHIP 66

Thinking About Additive Comparison Situations 66
- Students and Teachers Think About the Problems 70
- Language Can Get Tricky 71

Extending to Negative Values 74
- Students and Teachers Think About the Problems 75
- A Note About Equations 76
- Knowing When to Add and When to Subtract 77
- Modeling the Difference When Working With Negative Values 80

Key Ideas 82

Try It Out! 83

Reflect 84

CHAPTER FIVE. EQUAL GROUPS MULTIPLICATION AND DIVISION: TWO FACTORS, DIFFERENT JOBS 86

Thinking About Equal Groups Situations 86
- Students and Teachers Think About the Problems 90
- Multiplier and Measure Factors 91
- Making Sense of Factors Less Than One 93
- A Progression of Equal Groups Understanding 98

Thinking About Equal Groups Division 101
- Students and Teachers Think About the Problems 101
- Partitive Division 102
- Measurement (Quotitive) Division 103
- The Unknowns in Partitive and Measurement Division 104
- Matching Models to Contexts 105
- Tracking the Unit Whole and the Referent Whole 107

Key Ideas 112

Try It Out! 113

Reflect 115

CHAPTER SIX. RATIOS AND RATES: DESCRIBING RELATIONSHIPS 116

Thinking About Problem Situations With Ratios and Rates 116
- Students and Teachers Think About the Problems 120
- Defining Terms 121

Four Contexts of Ratios and Proportional Reasoning Problems 123
- Associated Sets 124
- Well-Chunked Measures 125
- Stretchers and Shrinkers 129
- Part-To-Whole 132
- Translating the Five Representations: Try It Out 138

Key Ideas 141
Try It Out! 142
Reflect 143

CHAPTER SEVEN. MULTIPLICATIVE COMPARISON: MORE THAN SCALE FACTORS 144

Thinking About Multiplicative Comparison Situations 144
- Students and Teachers Think About the Problems 148
- Additive Versus Multiplicative Comparison 149
- Constant Ratio 150
- Absolute and Relative Change 151
- When Multiplication Makes a Quantity Smaller 152
- The Dilation Ratio 155

Extending to Negative Values 156
- Students and Teachers Think About the Problems 157
- Negative Measure Factors and Vectors 159
- Negative Scale Factors and Reflections 159

Key Ideas 163
Try It Out! 163
Reflect 165

CHAPTER EIGHT. AREA/ARRAY: TWO FACTORS, SAME JOB 166

Thinking About Area/Array Situations 166
- Students and Teachers Think About the Problems 170
- Symmetric Versus Asymmetric Multiplication 171
- Representing Fraction and Decimal Division 174

Extending to Negative Values 176
- Models *of* Thinking and Models *for* Thinking 177
- Array as a Mathematical Structure 177

Key Ideas 181
Try It Out! 182
Reflect 184

CHAPTER NINE. PROBABILITY AND CARTESIAN PRODUCTS: COMBINATORICS 186

Thinking About Combinatorics Situations 186
- Students and Teachers Think About the Problems 190
- Understanding of Probability Is Based in Intuition 191
- Experiencing Experimental Outcomes in Game Format 192

Analyzing and Making Sense of Experimental Outcomes 194
Building Intuition With Games 196
Using Models to Predict Number of Outcomes 199
The Path Toward Systematic Thinking 199
Key Ideas 202
Try It Out! 203
Reflect 204

CHAPTER TEN. CHANGING HOW YOU TEACH WORD PROBLEMS 205

Getting Into the Mathematizing Sandbox 205
Eight Shifts in Instruction for Building Students' Problem-Solving Skills 207
Do Word Problems for Sense-Making 207
Treat Context and Computation Separately 208
Create More and Varied Representations 208
Explore All the Work Operations Can Do 209
Add Operation Sense Routines to the School Day 210
Offer Students Experiences With a Variety of Problem Situations 210
Listen to Students and Be Curious 211
Make Time for Mathematizing in the Sandbox 211
Guidance for Moving Forward: FAQs 212

Appendix 216

References 220

Index 224

online resources

Visit the companion website at
http://resources.corwin.com/problemsolving6-8
for downloadable resources.

List of Videos

Video 2.1 – Crocheting a Scarf With a Number Line 19
Video 2.2 – Collecting Recycling With Fraction Bars 19
Video 2.3 – Using Flour to Thicken With a Number Line 23
Video 2.4 – Andrea Earns Money With Play Money 25
Video 2.5 – John Spends Money With a Number Line 25
Video 2.6 – Submarine Descending With a Diagram 30
Video 2.7 – Balloon Descending With an Equation 30

Video 3.1 – Eating Licorice With Fraction Bars 48
Video 3.2 – Red and Black Licorice Think Aloud 48
Video 3.3 – Adding Angles With a Protractor 51
Video 3.4 – Finding a Missing Angle Measure With a Bar Model 51
Video 3.5 – Calcium Charge With Two-Color Counters 55
Video 3.6 – Chlorine Charge With Two-Color Counters 55
Video 3.7 – Lawn Mowing Business With a Profit & Loss Statement 59
Video 3.8 – Lawn Mowing Business With a Number Line 59

Video 4.1 – Money Earned Mowing With a Number Line 70
Video 4.2 – Farmer's Market With Bar Model 70
Video 4.3 – Dallas Temperatures With a Bar Model 75
Video 4.4 – Chicago Temperatures With a Number Line 75
Video 4.5 – Snapping Turtle With Anchor-Jump Strategy 81

Video 5.1 – Bagging Rice With Tally Marks 90
Video 5.2 – Packing Baby Formula With Base 10 Blocks 90
Video 5.3 – Slime With a Number Line 94
Video 5.4 – Business Ownership With a Ratio Table 94
Video 5.5 – Partitive Division With Folded Paper 102
Video 5.6 – Measurement Division With a Ratio Table 102
Video 5.7 – Sharing Dog Food With Color Tiles 109
Video 5.8 – Filling Dog Water Bowls With Fraction Tiles 109

Video 6.1 – Distributing Building Bricks With a Diagram 120
Video 6.2 – Travel Distance and Time With a Bar Model 120
Video 6.3 – Cost of Cups With a Graph 126
Video 6.4 – Flag Models With Graph Paper 131
Video 6.5 – Lesh Model Connections/Translations for Juice Problem 137

Video 7.1 – Gas Costs With a Bar Model 148
Video 7.2 – Distance to School on a Number Line 148
Video 7.3 – Scaling Up a Mural to a Cafeteria Wall 153
Video 7.4 – Scaling Down a Mural to a Paper Grid 153
Video 7.5 – Losing Half the Yards on a Field Diagram 158
Video 7.6 – Losing Twice the Yards on a Number Line 158

Video 8.1 – Carpet Length With a Grid Diagram 170
Video 8.2 – Maple Syrup With a Drawing 170
Video 8.3 – Fancy Paper With an Area Model 175

Video 9.1 – Rolling a Die With a Frequency Table 190
Video 9.2 – Flipping a Coin With a List of Outcomes 190
Video 9.3 – Sum of Two Dice With a Chart 198
Video 9.4 – Sum of Two Dice With a Tree Diagram 198
Video 9.5 – Possible Sandwiches With a List 201
Video 9.6 – Possible Sandwiches With a Tree Diagram 201

Preface

The three of us (Kim, Sara, and Linda) spend a lot of time with teachers, talking about how students are successful and what challenges they face. Over and over, we hear that students struggle with problem solving, especially with word problems. We see challenges expressed in teacher comments on Twitter, on Facebook, on community sites like MyNCTM, in conference sessions, even in the news and on other social media. We hear this frustration from parents, from principals, and from math coaches across the country and internationally. We've written this book for classroom teachers and coaches who want to help their students have a more successful and meaningful approach to problem solving. If the approaches you have tried, such as using key words, or even using reading strategies to help students comprehend the problem, have yielded only spotty or unsatisfying results, this book is for you. If your students compute naked numbers efficiently, but when faced with a word problem they seem to pull numbers at random from problems and end up successfully calculating the wrong equations, this book is for you. If your students have ever looked at an equation and wondered how it fits the problem, this book is for you. We've written this book for all the teachers whose students look at a word problem and say, "I just don't get what they want me to do!"

How This Book Can Help You

Would you be surprised to know that every addition word problem you can think of can be classified into one of four categories? It's true! The same is true for every subtraction problem. Multiplication and division are a bit more complicated, but not as much as you might think. The hard part about a word problem isn't in using the operations (+ – × ÷) to compute an answer but, rather, it's in figuring out which operation to use in a problem and why. Once you understand the four kinds of addition problem types and can recognize them in a problem's story, the puzzle pieces can start to come together. We don't mean to oversimplify the learning that needs to take place, because it isn't simple, but we want you to know that there is something you can do to help students learn to tackle word problems productively. More important, we want students to tackle real problems that interest them and learn more mathematics as they do so.

> *Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.*
>
> —*Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000, p. 52)*

There is no magic elixir to solve the problem of word problems. This book shares our approach, which focuses on helping children mine the problem to uncover the underlying mathematics. Much needs to happen between the reading comprehension and computation stages of the problem-solving process. Yes, students need support to read and comprehend the words, context, and language of the problem. Yes, they need to know how to compute the answer. But there is a whole middle ground of exploration and understanding that students often rush through, where they could instead turn what they read into a solvable mathematical story and apply their operation sense to solve the puzzle. This is where we so often see a gap. We've written this book to fill that gap. We want students to see their world mathematically and to know that mathematics can help them solve real problems. This is bigger (and more important) than passing a test.

To that end, this book is about problem solving. It's about deciphering the kinds of word problems you see in normal, everyday lessons in classrooms like yours. It's about the kinds of problems that are placed at the end of the lessons in your textbooks, the ones that your kids skip because they don't know what to do. It's about word problems. Story problems. Make-sense-of-the-math, practice-a-skill, not-quite-mathematical-modeling problems. Sometimes these problems can seem artificial or contrived, but their straightforward simplicity is also a strength because they target the mathematical thinking we want students to develop. Wrestling productively with word problems can lay a foundation for the more complex and open-ended problems students will also encounter. These problems also have the potential to *do* and *be* more in their own right when students can build the mathematical models that solve problems that interest them.

> *Solutions to these (routine word) problems, particularly the solutions of younger children, do in fact involve real problem-solving behavior… . Word problems can provide insights into the development of more complex problem-solving abilities.*
>
> *(Carpenter, 1985, p. 17)*

How to Use This Book

As a reader, you'll get the most out of this book if you dig in and do the mathematics along the way. We've given you space to restate problems and draw pictures of your thinking with a focus on the mathematics. You'll also find a collection of manipulatives helpful as you work through the book. Gather some counters, some base 10 blocks, and a fraction tool or two to use as aids while you solve problems. Think about how your students can use them too. Even in the middle grades, pictures, manipulatives, number lines, and bar models can hold an idea in place right in front of you so that you can think about it more deeply. With manipulatives, you can do even more. You can make a change more quickly and easily: Manipulatives allow students to act out what happens in a problem, and to use attributes like color and size to highlight features of a problem. As you'll see, the best tool for the job depends on the problem. There will be plenty of examples for you to explore.

There are places in the book where we've suggested you stop and talk with your colleagues. If you're reading the book as part of a professional learning community, plan your discussions around these stopping points. You'll find that there are plenty of opportunities for conversations about student thinking and the operations. But you're never truly reading alone! Throughout each chapter you'll find work samples inspired by student work. Several times in each chapter you'll also find teacher commentary on the work samples. These comments are honest. Sometimes the teachers are bewildered, and sometimes the teachers are excited by what they see. Let these teacher voices be your companions as you tackle the new ideas. Use the teacher comments and the thinking breaks as reminders to take a pause and extend your own ideas a little further. The end of each chapter also has exercises and reflection questions that will help you and your colleagues connect what you've learned to your own classrooms.

After each chapter we also suggest that you look at the problems in your textbook and categorize them—not just for practice recognizing the structures you'll soon learn about, but also to evaluate how much exposure your students are getting to the full range of problem types. If you discover that your textbook does not present enough variety, this book will give you the tools needed to make adjustments.

We recognize that many of the problems shared in this book will be unfamiliar contexts for your students. If you find yourself thinking that your students will not understand a problem context, we invite you take a moment to explore the new context and make sense of it. Or, change the problem! Make sense of the problem situation yourself so that you know what mathematical features are important and then change the details. Even better, invite your students to craft and pose their own meaningful word problems to solve. With your new understanding of the problem situations, you will have all the tools you need to guide your students.

We have to be honest. The ideas in this book may challenge your current understandings of some mathematical ideas. At times, we will ask you to look at something you have been doing since you were ten or eleven years old and revisit it with new eyes. This may cause some disequilibrium, and it may be uncomfortable at first. It's as if we are asking you to walk but by switching the foot you lead with. (Try it! It's not easy!) When the familiar becomes unfamiliar, we encourage you to take a deep breath, trust us, and lead with the other foot. We'll get you there. Here's to lifelong learning!

Acknowledgments

You can't write a book like this in a vacuum. We have met and worked with countless educators over the years and have discussed the ideas in this book with them. You know who you are and we hope you hear your voice in these pages. Thank you for your contribution to these thoughts and ideas. We credit you and appreciate you deeply.

We would also like to thank Erin Null, who started with one vision of this book, received a draft of another interpretation, and worked with us to land on the third—and we think best—vision. Your patience, support, and diligence pushed us continually forward. We also extend a profound debt of gratitude to Paula Stacey, who asked us the questions we needed to answer in a way that made the manuscript better. As we finish the series of books, we recognize Amy Marks, Tori Mirsadjadi, Jessica Vidal, and the entire Corwin team for helping to keep our work consistent and presenting it so carefully.

Thank you also to Debbie Thomas, Barbara Dougherty, Kimberly Rimbey, Julie McNamara, Linda Levi, Terrie Galanti, Johnny Lott, and Jeff Shih, whose thoughtful comments challenged us to know better and do better. We are indebted to you for your positive feedback, but even more so for your constructive criticism. Any flaws that remain are ours alone.

I would first like to thank Linda and Sara for trusting me enough to invite me to join this project. I appreciate your confidence, but more than anything, I appreciate your friendship. I also want to thank Dr. Megan Murray of the University of Hull, who introduced me to the idea that addition and subtraction problems weren't all the same. To the staff of GBW, Julie, Pat, Kim and her students, and Michelle and her friend the second-grade teacher: Thank you for sharing your time and brilliant students who have helped us collect and interpret their thinking from many different angles. I want to thank my daughter Cassandra and her friends for fielding random questions about math, chemistry, and physics at all hours of the day and night. The book is better because of your guidance. Finally, thank you to my husband, Greg, who watched me take the big leap of writing three books and never questioned my sanity.

—Kim

Thanks to Linda for starting us down this road with rich conversations. Thanks to Kim for coming on this journey with us. I appreciate your knowledge, your experience, your care, and your friendship. Thanks to Margie Mason, who first brought me into the world of mathematics education, and to all my friends and colleagues in this community, including those at ETA hand2mind and ORIGO Education, who have encouraged and supported me along the way. Thank you to the teachers who came before me, particularly my mother and grandmother, for showing me that learning is important and good teaching is invaluable. And thanks to Bill, for loving and supporting me always.

—Sara

Writing a book is always a challenge! While it seems that writing on a topic you feel passionate about should be easier, it is actually a bigger challenge because you want to get it right. I thank Kim and Sara for their vision and our many long conversations. I learned so much from both of you. I want to thank the elementary teachers and coaches with whom I work who challenge my thinking and force me to make ideas clearer. I thank my colleagues Ruth Harbin Miles, Annemarie Newhouse, and Jerry Moreno, whose friendship I value and who make this career a joy.

—Linda

Publisher's Acknowledgments

Corwin gratefully acknowledges the contributions of the following reviewers:

Kevin Dykema
Middle School Math Teacher
Mattawan Middle School
Mattawan, MI

Julie McNamara
Assistant Professor of Mathematics Education
California State University, East Bay
Hayward, CA

Kimberly Rimbey
Executive Director of Curriculum, Instruction, and Assessment
Buckeye Elementary School District
Buckeye, AZ

About the Authors

Kimberly Morrow-Leong is an adjunct instructor at George Mason University in Fairfax, Virginia, and a consultant for Math Solutions. She is a former grade 5–9 classroom teacher, researcher at American Institutes for Research, K–8 mathematics coach, and coordinator of elementary professional development for the National Council of Teachers of Mathematics (NCTM). She recently completed an elected term as vice president and 2018 program chair for NCSM, Leadership in Mathematics Education. She holds a BA in French language and a master's degree in linguistics (TESOL). She also holds an MEd and PhD in mathematics education leadership from George Mason University. Kim is the 2009 recipient of the Presidential Award for Excellence in Mathematics and Science Teaching (PAEMST) from Virginia. She is happiest when working with teachers and students, putting pencils down and getting messy with manipulatives!

Sara Delano Moore currently serves as director of professional learning and chair of the Mathematics Advisory Board at ORIGO Education. A fourth-generation educator, Sara's work emphasizes the power of deep understanding and multiple representations for learning mathematics. Her interests include building conceptual understanding to support procedural fluency and applications, incorporating engaging and high-quality literature into mathematics and science instruction, and connecting mathematics with engineering design in meaningful ways. Prior to joining ORIGO Education, Sara served as a classroom teacher of mathematics and science in the elementary and middle grades, a mathematics teacher educator at the University of Kentucky, director of the Kentucky Center for Middle School Academic Achievement, and director of mathematics and science at ETA hand2mind. She has authored numerous articles in professional journals and is a contributing author to *Visible Learning for Mathematics*. She has also coauthored the grades 3–5 and grades 6–8 volumes of the *Teaching Mathematics in the Visible Learning Classroom* series for Corwin Mathematics. Sara earned her BA in natural sciences from Johns Hopkins University, her MSt in general linguistics and comparative philology from the University of Oxford, and her PhD in educational psychology from the University of Virginia. She lives in Kent, Ohio.

Linda M. Gojak worked as an elementary mathematics specialist and classroom teacher for 28 years. She directed the Center for Mathematics and Science Education, Teaching, and Technology at John Carroll University for 16 years, providing support for districts and more than 10,000 teachers. Linda continues to work with K–8 mathematics teachers and coaches nationally and internationally. She is a recipient of the PAEMST from Ohio. She served as the president of NCTM, NCSM, and the Ohio Council of Teachers of Mathematics. Linda is the coauthor of three other books for Corwin Mathematics—*The Common Core Math Companion, K–2; The Common Core Math Companion, 6–8*; and *Visible Learning for Mathematics, Grades K–12*. Linda also wrote *Path to Problem Solving for Grades 3–6* (ETA hand2mind, 2008) and *What's Your Math Problem?* (Teacher Created Materials, 2011).

CHAPTER ONE

Introduction

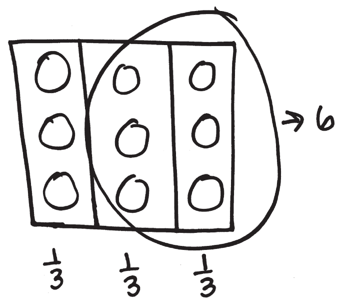
Why You Need to Teach Students to Mathematize

Imagine you are a new teacher. You are teaching eighth grade at a new school and are eager to get to know your students—their interests, skills, and how prepared they are to meet the challenges of eighth grade. You have just emerged from your teacher education program knowing various approaches you have seen modeled in classrooms and described in the literature, some of which you have tried with varying degrees of success. You aren't sure what approaches you want to use but are excited about challenging your students, introducing the rigor you have read so much about. But first, you need to know what your students can and can't do.

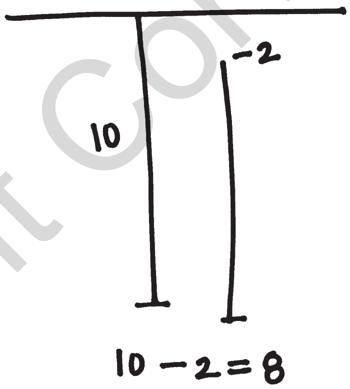
You decide to start with a couple of word problems, ones that involve relatively simple mathematical operations:

> *Mrs. King wanted her American history students to do a project about the Emancipation Proclamation. $\frac{2}{3}$ of the class chose to make podcasts. The other 9 students chose to create graphic novels. How many students are in Mrs. King's American history class?*
>
> *Armando started his descent into the cave. He was 10 feet down before he realized that he had forgotten to bring a flashlight. He climbed back up to the 2-foot mark to take the flashlight his friend handed to him. How many feet did he have to climb to get the flashlight?*

You circulate around the room, noting who draws pictures, who writes equations, and who uses the manipulatives you have put at the center of the table groups. While some students take their time, quite a few move quickly. Their hands go up, indicating they have solved the problems. As you check their work, one by one, you notice most of them got the first problem wrong, writing the equation $\frac{2}{3} \times 9 = 6$. Some even include a sentence saying, "6 students will do a podcast." Only one student in this group draws a picture. It looks like this:



Even though the second problem demands an understanding of integers, a potentially complicating feature, most of these same students arrive at the correct answer, despite the fact that they do not write a correct equation to go with it. They write the incorrect equation 10 – 2 = 8 and are generally able to find the correct answer of +8, representing an 8-foot climb toward the cave opening. Some write K C C above their equation. You notice that other students make a drawing to help them solve this problem. Their work looks something like this:



To learn more about how your students went wrong with the history assignment problem, you call them to your desk one by one and ask about their thinking. A pattern emerges quickly. All the students you talk to zeroed in on two key elements of the problem: (1) the portion of students who did a podcast ($\frac{2}{3}$) and (2) the word "of". One student tells you, "*Of* always means to multiply. I learned that a long time ago." Clearly, she wasn't the only student who read the word *of* and assumed she had to multiply by the only other number given in the problem. While a key word strategy led students astray in the first problem, visualizing the problem situation in the second problem led students to a correct answer, even if they were not able to write an accurate equation for the problem situation.

Problem-Solving Strategies Gone Wrong

In our work with teachers, we often see students being taught a list of "key words" that are linked to specific operations. Students are told, "Find the key word and you will know whether to add, subtract, multiply, or divide." Charts of key words often hang on classroom walls, even in middle school.

Key words are a strategy that works often enough that teachers continue to rely on them. They also seem to work well enough that *students* continue to rely on them. But as we saw in the history assignment problem, not only are key words not enough to solve a problem, they can also easily lead students to an incorrect operation or to an operation involving two numbers that aren't related (Karp, Bush, & Dougherty, 2014). As the history assignment problem reveals, different operations could successfully be called upon, depending on how the student approaches the problem:

1. A student could use subtraction $\left(1-\frac{2}{3}\right)$ to determine that $\frac{1}{3}$ of the students in the class made graphic novels.
2. A student could use division to find the number of students in the class, dividing the 9 students doing graphic novels by $\frac{1}{3}$ of the class to get 27, the number of students in the whole class. This could even be modeled using an array solution strategy like the one in the student's drawing seen earlier.

Let's return to your imaginary classroom. Having seen firsthand the limitations of key words—a strategy you had considered using—where do you begin? What instructional approach would you use? One of the students mentioned a strategy she likes called CUBES. If she learned it from an elementary teacher and still uses it, you wonder if it has value. Your student explains that CUBES has these steps:

Circle the numbers

Underline important information

Box the question

Eliminate unnecessary information

Solve and check

She tells you that her teachers walked students through the CUBES protocol using a "think-aloud" for word problems, sharing how they used the process to figure out what is important in the problem. That evening, as you settle down to plan, you decide to walk through some problems like the history assignment problem using CUBES. Circling the numbers is easy enough. You circle $\frac{2}{3}$ (podcast) and 9 (students), wondering briefly what students might do with the question "How many?" Perhaps it's too early to think of that for now.

Then you tackle "important information." What is important here in this problem? Maybe the fact that there are two different assignments. Certainly it's important to recognize that students do one of two kinds of history assignment. You box the question, but unfortunately the question doesn't help students connect $\frac{2}{3}$ to 9 with a single operation.

If you think this procedure has promise as a way to guide students through an initial reading of the problem, but leaves out how to help students develop a genuine understanding of the problem, you would be correct.

What is missing from procedural strategies such as CUBES and strategies such as key words, is—in a word—*mathematics* and the understanding of where it lives within the situation the problem is presenting. Rather than helping students learn and practice quick ways to enter a problem, we need to focus our instruction on helping them develop a deep understanding of the mathematical principles behind the operations and how they are expressed in the problem. They need to learn to *mathematize*.

What Is Mathematizing? Why Is It Important?

Mathematizing: The uniquely human act of modeling reality with the use of mathematical tools and representations.

Problem situation: The underlying mathematical action or relationship found in a variety of contexts. Often called "problem type" for short.

Solution: A description of the underlying problem situation along with the computational approach (or approaches) to finding an answer to the question.

Operation sense: Knowing and applying the full range of work for mathematical operations (for example, addition, subtraction, multiplication, and division).

Intuitive model of an operation: An intuitive model is "primitive," meaning that it is the earliest and strongest interpretation of what an operation, such as multiplication, can do. An intuitive model may not include all the ways that an operation can be used mathematically.

Mathematizing is the uniquely human process of constructing meaning in mathematics (from Freudenthal, as cited in Fosnot & Dolk, 2002). Meaning is constructed and expressed by a process of noticing, exploring, explaining, modeling, and convincing others of a mathematical argument. When we teach students to mathematize, we are essentially teaching them to take their initial focus off specific numbers and computations and put their focus squarely on the actions and relationships expressed in the problem, what we will refer to throughout this book as the **problem situation**. At the same time, we are helping students see how these various actions and relationships can be described mathematically and the different operations that can be used to express them. If students understand, for example, that equal-groups multiplication problems, as in the history assignment problem, may include knowing the whole or figuring out the whole from a portion, then they can learn where and how to apply an operator to numbers in the problem, in order to develop an appropriate equation and understand the context. If we look at problems this way, then finding a **solution** involves connecting the problem's context to its general kind of problem situation and to the operations that go with it. The rest of the road to the answer is computation.

Making accurate and meaningful connections between different problem situations and the operations that can fully express them requires **operation sense**. Students with a strong operation sense

- Understand and use a wide variety of models of operations beyond the basic and **intuitive models of operations** (Fischbein, Deri, Nello, & Marino, 1985)
- Use appropriate representations of actions or relationships strategically
- Apply their understanding of operations to any quantity, regardless of the class of number
- Can mathematize a situation, translating a contextual understanding into a variety of other mathematical representations

FOCUSING ON OPERATION SENSE

Many of us may assume that we have a strong operation sense. After all, the four operations are the backbone of the mathematics we were taught from day one in elementary school. We know how to add, subtract, multiply, and divide, don't we? Of course we do. But a closer look at current standards reveals nuances and relationships within these operations that many of us may not be aware of, may not fully understand, or may have internalized so well that we don't recognize we are applying an understanding of them every day when we ourselves mathematize problems both in real life and in the context of solving word problems. For example, current standards ask that students develop conceptual understanding and build procedural fluency in four kinds of addition/subtraction problems, including Add-To, Take-From, Compare, and what some call Put Together/Take Apart (we will refer to this category throughout the book as Part-Part-Whole). Multiplication and division have their own unique set of problem types as well. On the surface, the differences between such categories may not seem critical. But we argue that they are. Only by exploring these differences and the relationships they represent can students develop the solid operation sense that will allow them to understand and mathematize word problems and any other problems they are solving, whatever their grade level or the complexity of the problem. It does not mean that students should simply memorize the problem types. Instead they should have experience exploring all the different problem types through word problems and other situations. Operation sense is not simply a means to an end. It has value in helping students naturally come to see the world through a mathematical lens.

USING MATHEMATICAL REPRESENTATIONS

What would such instruction—instruction aimed at developing operation sense and learning how to mathematize word problems—look like? It would have a number of features. First, it would require that we give students time to focus and explore by doing fewer problems, making the ones they do count. Next, it would facilitate students becoming familiar with various ways to represent actions and relationships presented in a **problem context**. We tend to think of solving word problems as beginning with words and moving toward the use of variables and equations in a neat linear progression. But as most of us know, this isn't how problem solving works. It is an iterative and circular process, where students might try out different representations, including going back and rewording the problem, a process we call telling "the story" of the problem. The model that we offer in this book is based on this kind of active and expanded exploration using a full range of **mathematical representations**. Scholars who study mathematical modeling and problem solving identify five modes of representation: verbal, contextual, concrete, pictorial, and symbolic representations (Lesh, Post, & Behr, 1987).

Problem context: The specific setting for a word problem.

Mathematical representation: A depiction of a mathematical situation using one or more of these modes or tools: concrete objects, pictures, mathematical symbols, context, or language.

VERBAL A problem may start with any mode of representation, but a word problem is first presented verbally, typically in written form. After that, verbal representations can serve many uses as students work to understand the actions and relationships in the problem situation. Some examples are restating the problem; thinking aloud; describing the math operations in words rather than symbols; and augmenting and explaining visual and physical representations including graphs, drawings, base 10 blocks, fraction bars, or other concrete items.

CONTEXTUAL The contextual representation is simply the real-life situation that the problem describes. Prepackaged word problems are based on real life, as is the earlier flashlight problem, but alone they are not contextual. Asking students to create their own word problems based on real-life contexts will bring more meaning to the process and will reflect the purposes of mathematics in real life, such as when scientists, business analysts, and meteorologists mathematize contextual information in order to make predictions that benefit us all. This is a process called **mathematical modeling**, which Garfunkel and Montgomery (2019) define as the use of "mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena."

Mathematical modeling: A process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into real-world phenomena.

CONCRETE Using physical representations such as blocks, concrete objects, and real-world items (for example, money, measuring tools, or items to be measured such as beans, sand, or water), or acting out the problem in various ways, is called **modeling**. Such models often offer the closest and truest representation of the actions and relationships in a problem situation. Even problem situations where negative quantities are referenced can rely on physical models when a feature such as color or position of an object shows that the quantity should be interpreted as negative.

Modeling: Creating a physical representation of a problem situation.

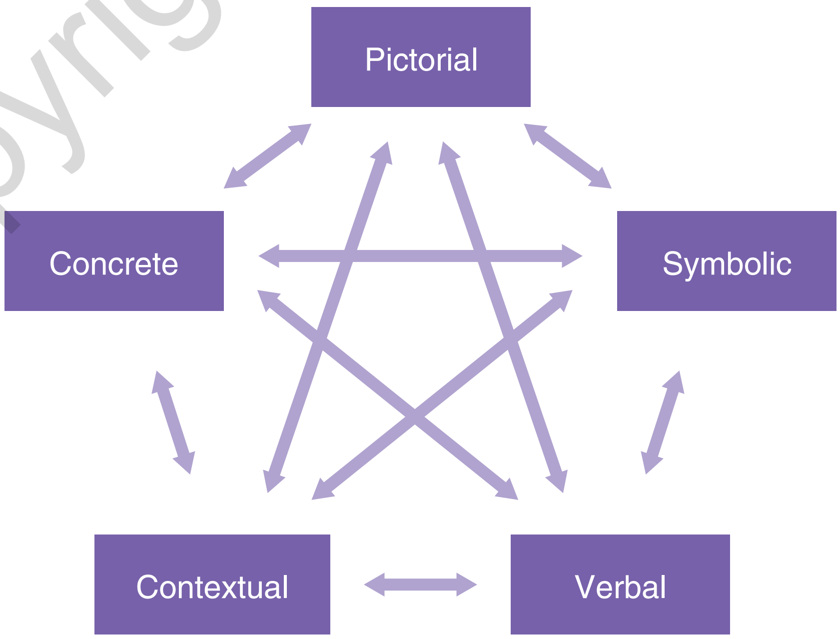
PICTORIAL Pictures and diagrams can illustrate and clarify the details of the actions and relationships in ways that words and even physical representations cannot. Using dots and sticks, bar models, arrows to show action, number lines, and various graphic organizers helps students see and conceptualize the nature of the actions and relationships.

SYMBOLIC Symbols can be operation signs (+, −, ×, ÷), relational signs (=, <, >), variables (typically expressed as x, y, a, b, etc.), or a wide variety of symbols used in middle school and in later mathematics (k, ∞, ϕ, π, etc.). Even though numerals are more familiar, they are also symbols representing values (2, 0.9, $\frac{1}{2}$, 1,000).

There are two things to know about representations that may be surprising. First, mathematics can be shared *only* through representations. As a matter of fact, it is impossible to share a mathematical idea with someone else without sharing it through a representation! If you write an equation, you have produced a *symbolic* representation. If you describe the idea, you have shared a *verbal* representation. Representations are not solely the manipulatives, graphs, pictures, and drawings of a mathematical idea: They are any mode that communicates a mathematical idea between people.

Second, the strength and value of learning to manipulate representations to explore and solve problems is rooted in their relationship to one another. In other words, the more students can learn to move deftly from one representation to another, translating and/or combining them to fully illustrate their understanding of a problem, the deeper will be their understanding of the operations. Figure 1.1 reveals this interdependence. The five modes of representation are all equally important and deeply interconnected, and they work synergistically. In the chapters that follow, you will see how bringing multiple and synergistic representations to the task of problem solving deepens understanding.

FIGURE 1.1 FIVE REPRESENTATIONS: A TRANSLATION MODEL



Source: Adapted from Lesh, Post, and Behr (1987).

Teaching Students to Mathematize

As we discussed earlier, learning to mathematize word problems to arrive at solutions requires time devoted to exploration of different representations with a focus on developing and drawing on a deep understanding of the operations. We recognize that this isn't always easy to achieve in a busy classroom, hence, the appeal of the strategies mentioned at the beginning of the chapter. But what we know from our work with teachers and our review of the research is that, although there are no shortcuts, structuring exploration to focus on actions and relationships is both essential and possible. Doing so requires three things:

1. Teachers draw on their own deep understanding of the operations and their relationship to different word problem situations to plan instruction.
2. Teachers use a model of problem solving that allows for deep exploration.
3. Teachers use a variety of word problems throughout their units and lessons, to introduce a topic and to give examples during instruction, not just as the "challenge" students complete at the end of the chapter.

In this book we address all three.

BUILDING YOUR UNDERSTANDING OF THE OPERATIONS AND RELATED PROBLEM SITUATIONS

The chapters that follow explore the different operations and the various kinds of word problems—or problem situations—that arise within each. To be sure that all the problems and situational contexts your students encounter are addressed, we drew on a number of sources, including the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), the work done by the Cognitively Guided Instruction projects (Carpenter, Fennema, & Franke, 1996), earlier research, and our own work with teachers to create tables, one for addition and subtraction situations (Figure 1.2) and another for multiplication and division situations (Figure 1.3). Our versions of the problem situation tables represent the language we have found to resonate the most with teachers and students as they make sense of the various problem types, while still accommodating the most comprehensive list of categories. These tables also appear in the Appendix at the end of the book.

NOTES

FIGURE 1.2 ADDITION AND SUBTRACTION PROBLEM SITUATIONS

ACTIVE SITUATIONS				
	Result Unknown	**Change Addend Unknown**	**Start Addend Unknown**	
Add-To	Paulo paid $4.53 for his sandwich. Then he paid $1.50 for a carton of milk to finish his lunch. How much was his lunch? $4.53 + 1.5 = x$ $4.53 = x - 1.5$	Paulo paid $4.53 for the sandwich in his lunch. Then he added a carton of milk to his tray to finish his lunch. The total for his lunch is $6.03. How much is a carton of milk? $4.53 + x = 6.03$ $4.53 = 6.03 - x$	Paulo added a sandwich to his tray. He added a carton of milk that cost $1.50 to his tray. With the sandwich and milk, his lunch cost $6.03. How much does the sandwich cost? $x + 1.5 = 6.03$ $6.03 - 1.5 = x$	
Take-From	There are 186 students in the 7th grade. 35 left to get ready to play in the band at the assembly. How many students are not in the band? $186 - 35 = x$ $35 + x = 186$	There are 186 students in the 7th grade. After the band students left class for the assembly, there were 151 students still in their classrooms. How many students are in the band? $186 - x = 151$ $x + 151 = 186$	35 band students left class to get ready to play in the assembly. There were 151 students left in the classrooms. How many students are in the 7th grade? $x - 35 = 151$ $35 + 151 = x$	
RELATIONSHIP (NONACTIVE) SITUATIONS				
	Total Unknown	**One Part Unknown**		**Both Parts Unknown**
Part-Part-Whole	The local ice cream shop asked customers to vote for their favorite new flavor of ice cream. 119 customers preferred mint chocolate chip ice cream. 37 preferred açai berry ice cream. How many customers voted? $119 + 37 = x$ $x - 119 = 37$	The local ice cream shop asked customers which new ice cream flavor they like best. 156 customers voted. 37 customers preferred açai berry ice cream. The rest voted for mint chocolate chip ice cream. How many customers voted for mint chocolate chip ice cream? $37 + x = 156$ $x = 156 - 37$		The local ice cream shop held a vote for their favorite new flavor of ice cream. The options were mint chocolate chip and açai berry ice cream. What are some possible combinations of votes? $x + y = 156$ $156 - x = y$
	Difference Unknown	**Greater Quantity Unknown**	**Lesser Quantity Unknown**	
Additive Comparison	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 71 cards. How many fewer cards does Roberto have than Jessie? $53 + x = 71$ $53 = 71 - x$	Jessie and Roberto both collect baseball cards. Roberto has 53 cards and Jessie has 18 more cards than Roberto. How many baseball cards does Jessie have? $53 + 18 = x$ $x - 18 = 53$	Jessie and Roberto both collect baseball cards. Jessie has 71 cards and Roberto has 18 fewer cards than Jessie. How many baseball cards does Roberto have? $71 - 18 = x$ $x + 18 = 71$	

online resources Situation charts are available for download at **http://resources.corwin.com/problemsolving6-8**

FIGURE 1.3 MULTIPLICATION AND DIVISION PROBLEM SITUATIONS

ASYMMETRIC (NONMATCHING) FACTORS				
	Product Unknown	**Multiplier (Number of Groups) Unknown**	**Measure (Group Size) Unknown**	
Equal Groups	Mayim has 8 vases to decorate the tables at her party. She cuts a ribbon $1\frac{3}{4}$ feet long to tie a bow around the vase. How many feet of ribbon does she need? $8 \times 1\frac{3}{4} = x$ $x \div 8 = 1\frac{3}{4}$	Mayim has some vases to decorate the tables at her party. She uses $1\frac{3}{4}$ feet of ribbon to tie a bow around each vase. If she uses 14 feet of ribbon, how many vases does she have? $x \times 1\frac{3}{4} = 14$ $x = 14 \div 1\frac{3}{4}$	Mayim uses 14 feet of ribbon to tie bows around the vases that decorate the tables at her party. If there are 8 vases, how many feet of ribbon are used on each vase? $8x = 14$ $14 \div 8 = x$	
	Product Unknown (*y*)	**(Unit) Rate Unknown (*k*)**	**Measure Unknown (*x*)**	
Ratio/Rate	Tom drove 60 miles per hour (on average) for 4 hours. How many miles did he travel? $4 \times 60 = y$ $\frac{y}{4} = 60$	Tom drove at the same speed (on average) during his entire 4 hour trip. He traveled a total of 240 miles. At what speed did he travel? $4k = 240$ $\frac{240}{4} = k$	Tom drove 60 miles per hour (on average) for all 240 miles of his trip. For how many hours did he travel? $60x = 240$ $\frac{240}{x} = 4$	
	Resulting Value Unknown	**Scale Factor (Times as many) Unknown**	**Original Value Unknown**	
Multiplicative Comparison	Armando's family is doing a puzzle this week that has 500 pieces. Next week's puzzle has 1.5 times as many pieces. How many pieces does next week's puzzle have? $500 \times 1.5 = x$ $x \div 1.5 = 500$	Sydney's middle school has 500 students. José's middle school has 750 students. How many times bigger than Sydney's school is José's school? $500x = 750$ $500 = 750 \div x$	Mrs. W didn't order enough tickets for the festival. Mr. D ordered 750 tickets. Mrs. W said, "You bought 1.5 times as many tickets as I did." How many tickets did Mrs. W order? $1.5 \times x = 750$ $750 \div x = 1.5$	
SYMMETRIC (MATCHING) FACTORS				
	Product Unknown	**One Dimension Unknown**		**Both Dimensions Unknown**
Area/Array	Mr. Bradley bought a new mat for the front entrance to the school. One side measured $3\frac{1}{3}$ feet and the other side measured 12 feet. How many square feet does the mat cover? $3\frac{1}{3} \times 12 = x$ $x \div 12 = 3\frac{1}{3}$	The 40 members of the student council lined up on the stage to take yearbook pictures. The first row included 8 students and the rest of the rows did the same. How many rows were there? $8x = 40$ $x = 40 \div 8$		Daniella was designing a foundation using graph paper. She started with 40 squares. How many units long and wide could the foundation be? $x \times y = 40$ $40 \div x = y$
	Sample Space (Total Outcomes) Unknown	**One Factor Unknown**		**Both Factors Unknown**
Combinatorics** (Probability and Cartesian Products)	Karen has 3 shirts and 7 pairs of pants. How many unique outfits can she make? $3 \times 7 = x$ $3 = x \div 7$	Evelyn says she can make 21 unique ice cream sundaes (1 scoop + 1 topping) using just ice cream flavors and toppings. If she has 3 flavors of ice cream, how many toppings does she have? $3y = 21$ $21 \div 3 = x$		Audrey can make 21 different fruit sodas using the soda mixing machine. How many different flavorings and sodas could there be? $xy = 21$ $x = 21 \div y$

(continued)

(continued)

*Equal Groups problems, in many cases, are special cases of a category that includes all ratio and rate problem situations. Distinguishing between the two categories is often a matter of interpretation. Since the Ratio/Rate category is a critically important piece of the middle school curriculum and beyond, the Ratio/Rate category is given its own row here.
**Combinatorics (Cartesian products) are typically not included in the table of multiplication and division problem situations. Since this is a category of problem situation addressed in middle school mathematics standards, it has been added to this table.

Note: These representations for the problem situations reflect our understanding based on a number of resources. These include the tables in the Common Core State Standards for mathematics (Common Core Standards Initiative, 2010); the problem situations as described in the Cognitively Guided Instruction research (Carpenter, Hiebert, & Moser, 1981), in Heller and Greeno (1979), and in Riley, Greeno, and Heller (1984); and other tools. See the Appendix and the companion website for a more detailed summary of the documents that informed our development of these tables.

These problem structures are seldom if ever identified in middle-grades standards. They are typically addressed in the early elementary grades as students master basic whole number operations, and taken as known from there. Many of the challenges middle-grades students have with word problems may be rooted in a lack of familiarity with the problem structures, so it is helpful for middle school math teachers to understand them and recognize them within a word problem. We open each chapter in this book with a look at the new problem situation structure with positive rational numbers (whole numbers, fractions, and decimals); the second part of each chapter examines the same structure when the full range of values (positive and negative) are included.

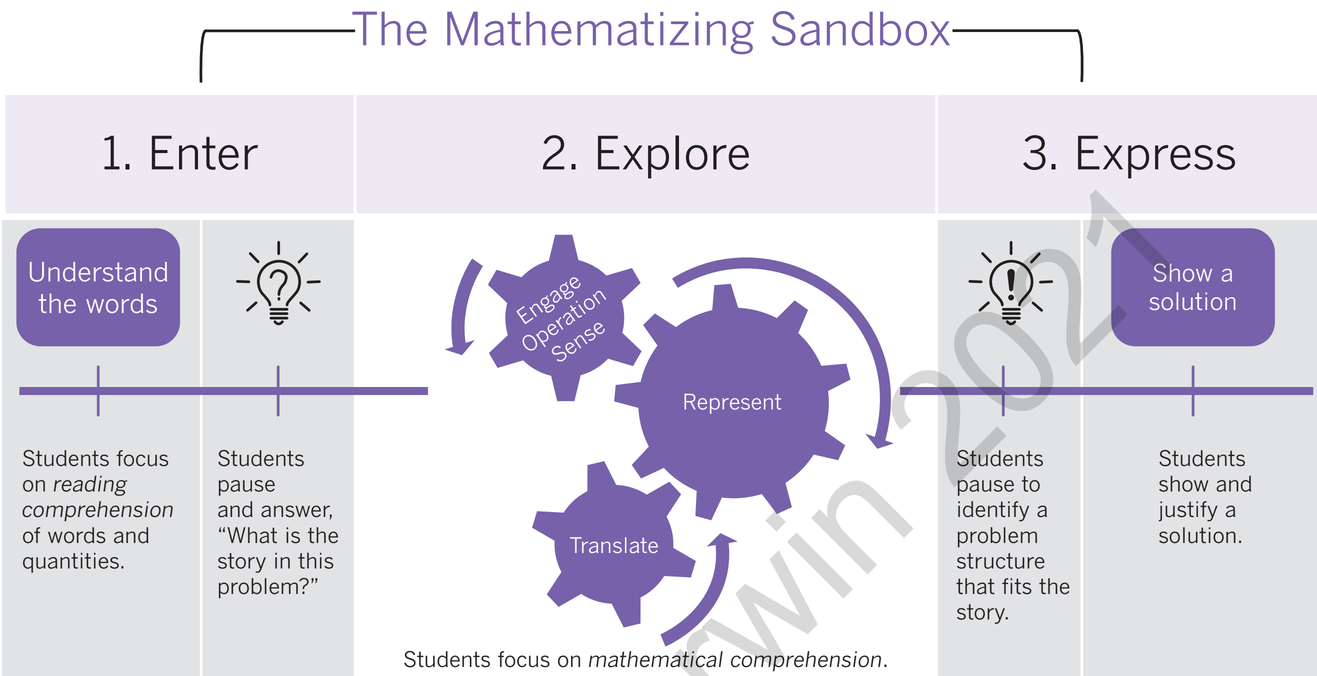
In the chapters—each of which corresponds to a particular problem situation and a row on one of the tables—we walk you through a problem-solving process that enhances your understanding of the operation and its relationship to the problem situation while modeling the kinds of questions and explorations that can be adapted to your instruction and used with your students. Our goal is *not* to have students memorize each of these problem types or learn specific procedures for each one. Rather, our goal is to help you enhance your understanding of the structures and make sure your students are exposed to and become familiar with them. This will support their efforts to solve word problems with understanding—through mathematizing.

In each chapter, you will have opportunities to stop and engage in your own problem solving in the workspace provided. We end each chapter with a summary of the key ideas for that problem situation and some additional practice that can also be translated to your instruction.

EXPLORING IN THE MATHEMATIZING SANDBOX: A PROBLEM-SOLVING MODEL

To guide your instruction and even enhance your own capacities for problem solving, we have developed a model for solving word problems that puts the emphasis squarely on learning to mathematize (Figure 1.4). The centerpiece of this model is what we call the "mathematizing sandbox," and we call it this for a reason. The sandbox is where children explore and learn through play. Exploring, experiencing, and experimenting by using different representations is vital not only to developing a strong operation sense but also to building comfort with the problem-solving process. Sometimes it is messy and slow, and we as teachers need to make room for it. We hope that this model will be your guide.

FIGURE 1.4 A MODEL FOR MATHEMATIZING WORD PROBLEMS



The mathematizing sandbox involves three steps and two pauses:

Step 1 (Enter): Students' first step is one of reading comprehension. Students must understand the words and context involved in the problem before they can really dive into mathematical understanding of the situation, context, quantities, or relationships between quantities in the problem.

Pause 1: This is a crucial moment when, rather than diving into an approach strategy, students make a conscious choice to look at the problem a different way, with a mind toward reasoning and sense-making about the *mathematical story* told by the problem or context. You will notice that we often suggest putting the problem in your own words as a way of making sense. This stage is critical for moving away from the "plucking and plugging" of numbers with no attention to meaning that we so often see (SanGiovanni & Milou, 2018).

Step 2 (Explore): We call this phase of problem solving "stepping into the mathematizing sandbox." This is the space in which students engage their operation sense and play with some of the different representations mentioned earlier, making translations between them to truly understand what is going on in the problem situation. What story is being told? What are we comparing, or what action is happening? What information do we have, and what are we trying to find out? This step is sometimes reflected in mnemonics-based strategies such as STAR (stop, think, act, review) or KWS (What do you know? What do you want to know? Solve it.) or Pólya's (1945) four steps to problem solving (understand, devise a plan, carry out a plan, look back) or even CUBES. But it can't be rushed or treated superficially. Giving adequate space to the Explore phase is essential to the understanding part of any strategic approach. This is where the cognitive sweet spot can be found, and this step is what the bulk of this book is about.

Pause 2: The exploration done in the mathematizing sandbox leads students to the "a-ha moment" when they can match what they see happening in the problem to a known problem situation (Figures 1.2 and 1.3). Understanding the most appropriate problem situation informs which operation(s) to use, but it also does so much more. It builds a solid foundation of operation sense.

Step 3 (Express): Here students leave the sandbox and are ready to express the story either symbolically or even in words, graphs, or pictures, having found a solution they are prepared to discuss and justify.

A Note About Negative Values

Negative rational number values represent multiple challenges for students. The shortcuts and rules that are often taught can feel nonsensical or random, and students may have internalized ideas about computation that are now challenged. For example, students may still believe that addition and multiplication always make things bigger. This is not necessarily true, and that realization is a big cognitive transition for students to make.

We know that integer computation is a challenging skill for many students to develop. It remains, even for some adults, a mystery of mathematics that equations like this one ($^{-}6 - {}^{-}8$) with so many signs expressing a negative value, still yields a positive 2. After all, how can subtraction and two negative numbers possibly yield a positive result? For that matter, why does a negative multiplied by a negative give a positive product? However, our focus in this book is not on computation strategies but, rather, on making sense of problem situations.

We firmly believe that if students reason about the problem situation, they can not only find a solution pathway, but they are more likely to understand where the answer comes from and why it's correct. Further, a deeper understanding of the structure of the problem situations and operations better prepares them to engage in mathematical modeling now as well as in future mathematics classes and into adulthood.

In each chapter, we will explore the problem situation first with fractions, decimals, and whole numbers. In the second half of each chapter, we introduce problem situations that include negative values. We also explore the symbols used in mathematics to describe a negative value. The negative symbol (–) actually has three different meanings (Stephan & Akyuz, 2012):

1. *Subtraction:* This symbol ($-x$), which children learn in elementary school, functions like a verb, an operator between the two values that come before and after the symbol.
2. *Less than zero:* In the middle grades, we introduce a symbol (^{-}x) that distinguishes a negative from a positive number. In this case, the symbol functions more like an adjective. For example, the symbol in front of $^{-}5$ describes a value that is 5 units less than zero. In contrast the symbol in front of $^{+}5$ describes a value 5 units greater than zero.
3. *The opposite:* This use of the negative symbol (^{-}x) conveys the idea of "the opposite," or the additive inverse. In this respect, it toggles back and forth between positive and negative values. Reading ^{-}x as "the opposite of x" instead of as "negative x"

communicates that ⁻x represents the additive inverse of x. If the value of x is already negative, students are often confused by the outcome. For example, when x is ⁻5, ⁻x is the additive inverse of ⁻5, or ⁺5. How can a number that appears negative (⁻x) have a positive value (⁺5)?

Distinguishing among these three different uses of the negative symbol may help students recognize them in context and help them be more deliberate in their own use. Conventions about the use of negative numbers are not intuitive for students (Whitacre et al., 2014). They may initially use values and signs (magnitude and direction) in ways that make sense to them but that may or may not correspond to standard conventions (Kidd, 2007). The flashlight problem at the beginning of this chapter is typical. The student's solution relied entirely on positive numbers and a subtraction operator to find the correct answer (10 – 2 = 8). This worked for the student likely because she recognized that the explorer never reached 0 to leave the cave. However, a more accurate equation for a problem situation that describes a descent and a climb out of a cave needs to include negative values to be accurate, as in $^{-}10 + x = {}^{-}2$. Does this matter? In this book we will make the case that it does matter. The incorrect equation given by this student may not be so much a "mistake" as it is a mistranslation of her understanding of the problem situation to a more accurate notation. We will return many more times to this idea of connecting the meaning of a problem situation to the various representations used to describe it.

Final Words Before You Dive In

We understand that your real life in a school and in your classroom puts innumerable demands on your time and energy as you work to address ambitious mathematics standards. Who has time to use manipulatives, draw pictures, and spend time writing about mathematics? Your students do! This is what meeting the new ambitious standards actually requires. It may feel like pressure to speed up and do more, but paradoxically, the way to build the knowledge and concepts that are currently described in the standards is by slowing down. Evidence gathered over the past 30 years indicates that an integrated and connected understanding of a wide variety of representations of mathematical ideas is one of the best tools in a student's toolbox (or sandbox!) for a deep and lasting understanding of mathematics (Leinwand et al., 2014). We hope that this book will be a valuable tool as you make or renew your commitment to teaching for greater understanding.