DEVELOPING FRACTIONS KNOWLEDGE

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Professional Learning and Three Grand Organizers for Arithmetic Instruction

Chapter 1 outlines ways in which the book can be used to support professional learning focused on strengthening instruction in arithmetic. In order to demonstrate this, three scenarios of professional learning will be included: (i) a mathematics coach working one-on-one to support a teacher in developing an instructional plan for her class; (ii) a school-based mathematics leader working with a team of teachers to revise their school's mathematics programme; (iii) a district-wide Math Recovery® Leader meeting with a team of intervention teachers to develop specialist knowledge aimed at advancing low-attaining students' fractions knowledge. Chapter 1 will also describe links from this text to major generic topics appearing in the current Mathematics Recovery Series, including guiding principles for instruction, domains of arithmetic knowledge and dimensions of mathematizing, such as **complexifying**, **distancing the instructional setting**, **formalizing**, **generalizing**, **notating** and **unitizing**.

Scenario 1: School-Based Mathematics Leader

Mr Phillips is a school-based mathematics leader working with a team of teachers to revise aspects of their school's mathematics programme. The Professional Learning Team (PLT) plans to meet initially for two hours, and then for one hour per week, for a period of six weeks to focus on the teaching of fractions in 4th and 5th grade. Their goal is to develop an instructional programme that (a) is more focused on arithmetical content than their current programme; and (b) enables teachers to take better account of students' current levels of fractions knowledge. Two members of the team have already developed a schedule of assessment tasks drawn from Chapters 3–6, and have administered the schedule individually to each of eight students covering a wide range of attainment levels. At their first meeting the team will review the video recordings of the assessment interviews and develop a simple means of coding students' responses to each of the assessment tasks. Extrapolating from the results of the assessment interviews, they will develop a set of 12 lessons to be taught over a six-week period, based on Chapter 4 (Fragmenting), Chapter 5 (Part–Whole Reasoning), Chapter 6 (Measuring with Unit Fractions) and Chapter 7 (Reversible Reasoning). Instructional plans for each lesson include: (a) a description of the main topic for the lesson; (b) instructional materials; (c) worksheets of relevant exercises incorporating a progression of difficulty; (d) the range of student responses to relevant assessment tasks; and (e) descriptions of common errors, difficulties and misconceptions. For each lesson, the instructional plan includes a 10-minute segment allowing for intensive, targeted intervention with a group of up to six students. This segment focuses on helping students to develop and consolidate their knowledge of the earlier topics of Fragmenting (Chapter 4) and Part–Whole Reasoning (Chapter 5).

Scenario 2: District-Wide Math Recovery® Leader

Ms Gomez is a district-wide Math Recovery® (MR) Leader. In the last two years she has trained two cohorts of 12 MR Intervention Specialists who work in schools across the district. This training has focused on teaching whole number arithmetic across grades K–4 and has involved the following three phases: Phase 1 consists of an initial professional learning programme of three to five days focusing on assessing and profiling students' arithmetical knowledge. Up to 12 students judged as likely to benefit from intensive intervention are individually administered a schedule of assessment tasks. The schedule of tasks has been developed from the sets of assessment tasks in Chapters 3–9. Each teacher's pre-assessment interviews are video-recorded for later analysis. Phase 2 consists of selecting up to four students, each of whom is taught individually for up to five 30-minute sessions per week, for teaching cycles of 12–15 weeks. Phase 2 also includes three or four on-going professional learning sessions during the period of the teaching cycle. Phase 3 involves administering one-on-one post-assessments to the 12 students who underwent the pre-assessment. Each teacher routinely video-records all of their teaching sessions as well as their pre- and post-assessment interviews. In all of the professional learning sessions, the participating teachers present case studies highlighting students' arithmetical strategies and progressions in learning. More detailed descriptions of the professional learning programme described above are available (Wright, 2000, 2003, 2008). This year Ms Gomez will again train two cohorts of Intervention Specialists, but for the first time one cohort will focus on the teaching of fractions at the 5th and 6th grade. In working with this new cohort, Ms Gomez will adopt the year-long professional development model that she has used for several years with other cohorts of teachers in her district.

Scenario 3: Mathematics Coach

Ms Liang is a mathematics coach working one-on-one to support a 5th grade teacher (Ms Koppel). The focus of the coaching is to use video-recorded, one-on-one assessments to advance Ms Koppel's knowledge of fractions pedagogy. Ms Liang meets with Ms Koppel to discuss their working together for a two-week period during which they will develop, administer and review a short schedule of assessment tasks adapted from the assessment tasks in Chapters 5 and 6. These tasks will be designed to assess students' mental actions relating to the fractions topics of partitioning and iterating. Using the assessment schedule, Ms Liang conducts one-on-one, video-recorded assessments with five students from Ms Koppel's class. The five students are selected as representative of a wide range of attainment levels in the learning of fractions. The purposes of the assessments are (a) to inform both coach and teacher of the range and nature of student responses to the assessment tasks and (b) to induct Ms Koppel into the process of using one-on-one assessments to gauge students' current levels of fractions knowledge. In their next meeting they review the video records using the assessment schedule to note students' responses to the tasks. Ms Liang takes the opportunity to highlight the interactive and inquiring nature of the assessment interview. The next day, Ms Koppel conducts one-on-one assessment interviews with a similar group of five students and in their next meeting they use the assessment schedule to review the video records of the second group of students, noting students' responses. As well, Ms Liang reviews Ms Koppel's conducting of the assessment interviews, noting the extent to which Ms Koppel successfully elicits valuable information about the fractions thinking of her students.

Three Grand Organizers for Arithmetic Instruction

In professional learning work with teachers focusing on whole number arithmetic, we developed three grand organizers (see Wright et al., 2006a, 2006b, 2012, 2015). These are: Guiding Principles for Instruction; Domains of Arithmetic Learning; and Dimensions of Mathematizing. In the following sections these grand organizers are described and extended to reasoning and arithmetic involving fractions.

1 Guiding Principles for Instruction

The authors of this book have conducted an extensive range of research and development projects focusing on mathematics pedagogy. Many of these projects involved working in close collaboration with teachers, schools and school systems to advance mathematics instruction. Below we set out nine guiding principles of mathematics instruction which aptly summarize the approach to fractions pedagogy that we advocate. In our collaborative research and development work, these principles have been applied extensively to guide the teaching of mathematics.

[1] The teaching approach is inquiry-based, that is, problem-based. Students routinely are engaged in thinking hard to solve fractions problems that for them are quite challenging.

[2] Teaching is informed by an initial, comprehensive assessment and on-going assessment through teaching. The latter refers to the teacher's informed understanding of students' current knowledge and problem-solving strategies, and continual revision of this understanding.

[3] Teaching is focused just beyond the ‘cutting-edge’ of students’ current knowledge.
[4] Teachers exercise their professional judgement in selecting from a range of teaching procedures each of which involves particular instructional settings and tasks, and varying this selection on the basis of on-going observations.
[5] The teacher understands students’ fractions strategies and deliberately engenders the development of more sophisticated strategies.
[6] Teaching involves intensive, on-going observation by the teacher and continual micro-adjusting or fine-tuning of teaching on the basis of that observation.
[7] Teaching supports and builds on students’ intuitive strategies and these are used as a basis for the development of written forms of mathematics that accord with students’ verbally-based and pictorially-based strategies.
[8] The teacher provides students with sufficient time to solve a given problem. Consequently students are frequently engaged in episodes that involve sustained thinking, reflection on their thinking and reflection on the results of that thinking.
[9] Students gain intrinsic satisfaction from their problem-solving, from their realization that they are making progress and from the verification methods they develop.

Each of these principles is now discussed in more detail.

Principle 1

> The teaching approach is inquiry-based, that is, problem-based. Students routinely are engaged in thinking hard to solve fractions problems that for them are quite challenging.

The inquiry-based approach to teaching mathematics is sometimes referred to as learning through problem-solving or problem-based learning. In this approach, the central learning activity for students is to solve tasks that constitute genuine problems – problems for which the students do not have a ready-made solution. What follows is that the issue of whether a particular task is appropriate as a genuine problem largely depends on the extent of their current knowledge.

Principle 2

> Teaching is informed by an initial, comprehensive assessment and on-going assessment through teaching. The latter refers to the teacher’s informed understanding of students’ current knowledge and problem-solving strategies, and continual revision of this understanding.

Assessment for providing specific and detailed information to inform instruction is the critical ingredient in our approach to teaching mathematics. Through assessment, teachers can develop a working model of students’ current knowledge of fractions. Thus it is essential to conduct a detailed assessment of their current knowledge, and to use the results of that assessment in designing instruction. As well, it is essential to revise and update one’s understanding of students’ knowledge through on-going, close observation of their responses to assessment tasks. The chapters that follow contain detailed descriptions of assessment tasks and notes on their use. These have the explicit purpose of informing the design of instruction. The second aspect of this principle, on-going assessment through observation and reflection, is equally as important as initial assessment and a teacher's understanding of a student's current knowledge can always be deepened.

Principle 3

> Teaching is focused just beyond the 'cutting-edge' of students' current knowledge.

This principle accords with Vygotsky's (1978: 84–91) notion of the zone of proximal development and Steffe's (1991) notion of the zone of potential construction; that is, instruction should be focused just beyond the student's current levels of knowledge in areas where that student is likely to learn successfully through sound teaching. This principle is very important in our focus on fractions instruction, and it highlights the importance of assessment to inform teaching. As well, this principle highlights the importance of teachers' evolving understanding of students, and of the idea that productive struggle is essential in learning. Assessment provides the teacher with a profile of students' knowledge and the teacher focuses instruction to provide opportunities for students to move beyond their current levels of knowledge.

Principle 4

> Teachers exercise their professional judgement in selecting from a range of teaching procedures each of which involves particular instructional settings and tasks, and varying this selection on the basis of on-going observations.

This principle highlights the need to develop a range of instructional procedures and to understand the role of each procedure, in terms of its potential to bring about advancements in students' current knowledge. The chapters that follow contain detailed descriptions of instructional activities that can be used to develop appropriate teaching procedures. Also, the assessment tasks provided in each chapter constitute an additional source of instructional procedures because the tasks are easily adapted for instruction.

Principle 5

> The teacher understands students' fractions strategies and deliberately engenders the development of more sophisticated strategies.

This principle highlights the need for teachers to have a working model of students' knowledge of fractions and the ways in which their knowledge typically progresses. Each of the chapters that follow provides a detailed overview of the development of aspects of fractions knowledge. Our belief is that, through reading, observing and reflecting, in conjunction with their teaching practice, teachers can significantly develop their knowledge of fractions pedagogy.

Principle 6

> Teaching involves intensive, on-going observation by the teacher and continual micro-adjusting or fine-tuning of teaching on the basis of that observation.

This principle highlights the importance of observational assessment in determining students' specific learning needs, and the need for this assessment to be on-going and to lead to action, that is, the fine-tuning of instruction on the basis of on-going assessment.

Principle 7

> Teaching supports and builds on students' intuitive strategies and these are used as a basis for the development of written forms of mathematics that accord with students' verbally-based and pictorially-based strategies.

This principle highlights that students express and develop their fractions knowledge, at least initially, via verbal expressions and images. That is, they develop their knowledge via images if they are supported to work with and develop their ideas with pictures and manipulatives that open possibilities for them to visualize quantities and verbalize their findings.

Principle 8

> The teacher provides students with sufficient time to solve a given problem. Consequently students are frequently engaged in episodes that involve sustained thinking, reflection on their thinking and reflection on the results of that thinking.

In our research and development work in mathematics pedagogy, we emphasize the importance of sustained thinking and reflection for the learning of mathematics. The topic of fractions is well suited to significant problem-solving by students. This problem-solving and the mental processes of thinking hard and reflecting during problem-solving are, we believe, a fundamental aspect of learning fractions.

Principle 9

> Students gain intrinsic satisfaction from their problem-solving, from their realization that they are making progress and from the verification methods they develop.

This principle relates to Principle 8. Our experience in working closely with teachers and students for many years on the topics of whole number arithmetic and fractions is that when young students work hard at problem-solving and that problem-solving is successful, this is typically a very positive experience for the learner. To go further, we argue that this kind of learning constitutes a kind of cognitive therapy, having intrinsic rewards beyond teacher affirmation and peer recognition.

2 Domains of Arithmetic Knowledge

In our research and development work related to mathematics pedagogy, we have found it helpful to organize instruction in whole number arithmetic into a framework of bands and domains. In this book we extend this framework to include fractions pedagogy (see Table 1.1).

Table 1.1 Knowledge of whole number arithmetic

Band 1 – Very early number
Band 2 – Early number
Number words and numerals
Early counting, addition and subtraction
Structuring numbers 1 to 10
Band 3 – Middle number
Advanced number words and numerals
Structuring numbers 1 to 20
Conceptual place value
Addition and subtraction to 100
Multiplicative strategies
Multiplication and division basic facts
Knowledge of fractions
Band 4 – Fractions as solely part–whole concepts
Band 5 – Fractions as early measures
Band 6 – Fractions as fractional numbers

This framework of domains of knowledge provides a basis for extension from whole number arithmetic to the arithmetic of fractions. Each of these bands and domains is briefly described below. More detailed descriptions of these domains are available (e.g. Wright et al., 2012, 2015).

Band 1 – Very Early Number

In the first three or four years of life children develop nascent number knowledge. This earliest domain of number knowledge can be organized into subdomains. First, children hear number words and begin to organize them as a special set of words that often occur in a sequence, first forwards and then backwards as well. Second, children begin to learn to name or read numerals – initially this mainly involves one-digit numerals (e.g. 5, 8, 3) and is gradually extended beyond 10. Third, children acquire early notions of counting – to count is to say the words from one to four, for example while making pointing actions, and emphasizing the last number word, but the points might not necessarily be in one-to-one coordination with the number words. Fourth, children also begin to ascribe number to the simple configurations that occur on dice, dominoes and so on. All of this emerging knowledge provides an excellent basis for developing more advanced knowledge.

Band 2 – Early Number

We use the term ‘early number’ to label the arithmetic knowledge that children typically develop in the first two or three years of school (Table 1.1). We organize this knowledge into three domains and each of these can be linked to the subdomains of very early number (see above). These domains are (a) number words and numerals; (b) early counting, addition and subtraction; and (c) **structuring numbers** 1 to 10. Our co-series book entitled *Teaching Number* *in the Classroom with 4–8 year olds* (2nd edition, 2014) addresses these domains in great detail, and so only brief descriptions are provided here (see also Wright, 2013). The domain of number words and numerals includes children's learning of forward number word sequences, the number word after a given number word, backward number word sequences, the number word before a given number word, and learning to read and write numerals. The domain of early counting, addition and subtraction includes children's use of counting, typically by ones, in increasingly sophisticated ways to solve tasks involving how many items in a collection, how many in all in the case of two collections, how many remaining and so on. In this domain children solve counting-based tasks typically in the ranges 1 to 10 and 1 to 20. Important to realize is the clear distinction between the two domains just described (Wright, 2013). Finally, the domain of **structuring numbers** 1 to 10 involves developing early arithmetical knowledge that does not involve counting-by-ones. This includes topics such as: (a) partitioning 5, partitioning 10, the doubles of numbers in the range 1 to 5; and (b) using that knowledge to solve addition and subtraction tasks in the range 1 to 10.

Band 3 – Middle Number

We use the term 'middle number' to label the whole number arithmetic knowledge that children typically develop in the range 2nd to 5th grade (Table 1.1). We organize this knowledge into six domains and these can be linked to the domains of early number (Table 1.1). These domains are (a) Advanced Number Words and Numerals; (b) **Structuring Numbers** 1 to 20; (c) Conceptual Place Value; (d) Addition and Subtraction to 100; (e) Multiplicative Strategies; and (f) Multiplication and Division Basic Facts. Our co-series book entitled *Developing Number Knowledge: Assessment, Teaching and Intervention with 7–11 year olds* (2012) addresses these domains in great detail, and so only brief descriptions are provided here. The domains of advanced number words and numerals and **Structuring Numbers** 1 to 20 extend the corresponding domains in the Early Number Band. The domain of Conceptual Place Value encompasses a novel approach to place value instruction. This domain focuses on developing increasingly sophisticated strategies for incrementing and decrementing by units of 1s, 10s, 100s and so on. Sound knowledge of this domain provides an important basis for the domain of Addition and Subtraction to 100, that is, developing advanced strategies for mental addition and subtraction, such as the jump and split kinds of strategies (Wright et al., 2012). The last mentioned domain also includes learning to add and subtract to and from a decuple without counting by ones. The domains of Multiplicative Strategies and Multiplication and Division Basic Facts include: (a) learning to structure numbers multiplicatively, for example 24 is 4 × 6, 12 × 2 and so on; (b) developing increasingly sophisticated strategies for multiplication in the range 1–100 and beyond; and (c) meaningful habituation of the basic facts of multiplication and division (Wright et al., 2012). Collectively, the domains of Middle Number provide the basis for extending whole number arithmetic knowledge to fractions knowledge, and the latter is the focus of subsequent chapters of this book.

Band 4 – Fractions as Solely Part–Whole Concepts

Band 4 encompasses students who have constructed Parts Within Wholes and Parts Out of Wholes Fraction Schemes where fractions are determined based on the number of equal parts in the fraction and the whole. Band 4 is the focus of Chapters 4 and 5.

Band 5 – Fractions as Early Measures

Band 5 encompasses students who are beginning to understand fractions as sizes relative to the whole. These students are moving away from solely part–whole concepts, but they have not yet fully developed fractions as measures or numbers. Band 5 is the focus of Chapters 6 and 7, and it is also addressed in Chapters 9–12.

Band 6 – Fractions as Numbers

Band 6 encompasses students who have constructed Iterative Fraction Schemes and, thus, conceive of fractions truly as numbers and as extensive quantities (i.e. measures). Band 6 is the focus of Chapter 8 and is also addressed in Chapters 9–12.

3 Dimensions of Mathematizing

Progressive mathematization refers to progression to new learning that is mathematically more sophisticated than current learning (Gravemeijer et al., 2000). In our work focusing on developing whole number arithmetic knowledge, we developed a model of 11 dimensions of mathematizing:

1. Complexifying
2. Decimalizing numbers
3. Distancing the instructional setting
4. Extending the range of numbers
5. Formalizing
6. Generalizing
7. Grounded habituation
8. Notating
9. Refining computational strategies
10. Structuring numbers
11. Unitizing

Our co-series book entitled *Developing Number Knowledge: Assessment, Teaching and Intervention with 7–11 year olds* (2012) addresses these dimensions in detail (see also Ellemor-Collins and Wright, 2011). In this book we describe these dimensions from the perspectives of whole number arithmetic and students' developing fractions knowledge. As well, examples of mathematizing are highlighted in the instructional activities.

Complexifying

Complexifying refers to making an instructional situation more complex arithmetically. Examples include the following: progressing from comparing unit fractions to comparing non-unit fractions; progressing from adding or multiplying proper fractions to adding or multiplying improper fractions; progressing from comparing improper fractions just larger than one to improper fractions much larger than multiple wholes; and progressing from dividing whole numbers by fractions to dividing mixed numbers by fractions.

Decimalizing

Decimalizing refers to coming to know the base-ten structure of the numeration system. Thus mathematizing by **decimalizing** is a particular form of mathematizing by structuring (see below). Initially, students begin to realize the additive structure of two-digit numbers – a decuple plus a number in the range 0–9 (e.g. 46 consists of 40 and 6). This initial **decimalizing** of numbers is extended to a structure that is both additive and multiplicative (e.g. 346 consists of $3 \times 100 + 4 \times 10 + 6 \times 1$). At much the same time, this knowledge is extended to three- and four-digit numbers and beyond. **Decimalizing** also involves learning the system of decimal fractions and the links between decimal fractions and common fractions. This can lead to the realizations that every common fraction can be expressed as a terminating or repeating decimal and vice versa. Finally, arithmetic involving common fractions provides the important basis for progressing from arithmetic to algebra (see Chapter 13).

Distancing the Instructional Setting

We use the term '*instructional setting*' to describe the materials that teachers use to support students' reasoning about fractions. In some instructional situations it can be helpful to screen the materials and perhaps also to screen the notations students generate. The process of screening can support students' visualization associated with reasoning about fractions, which in turn can support their development of fractions knowledge. For example, a teacher asks students to visualize taking a unit fraction of a unit fraction after having made drawings.

Extending the Range of Numbers

When presenting arithmetical tasks to students, there is always scope to move beyond an initial range of numbers. In the case of addition with whole numbers for example, students' learning focuses initially on adding two numbers in the range 1–10, and this can extend to 20, to 100 and so on. In the case of fractions instruction, the initial step from addition involving whole numbers to addition involving simple fractions is an example of **extending the range of numbers**. Within fractions, instruction can extend from unit fractions to non-unit proper fractions and further to improper fractions.

Formalizing

Formalizing refers to students' progression from initial ways of reasoning and solving problems involving fractions to forms of reasoning typical of expert adults. This could involve students' progressing from intuitive procedures to formal algorithms when operating on fractions. For example, in adding two fractions, say $\frac{2}{3}$ and $\frac{3}{5}$, students might need to rely on drawings, at first, in order to find a common measurement unit $(\frac{1}{15})$ – partitioning each of the thirds into five equal parts and partitioning each of the fifths into three equal parts. **Formalizing** would involve representing that activity in a number sentence: $\frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$.

Generalizing

Generalizing refers to progressing from arithmetical to algebraic reasoning about fractions. Examples include **generalizing** in situations that involve comparing unit fractions according to the size of the denominator; describing a process for generating fractions equivalent to a fraction expressed in lowest terms; and describing a process for multiplication or division involving two fractions.

Grounded Habituation

A student who, when asked '8 plus 9', almost immediately answers '17', and similarly answers other additions in the range 1 to 20, has *habituated* addition facts. When asked to justify the answer the student might say, 'I know 8 + 8 makes 16 and I need to add one more'. Thus a student who has **grounded habituation** not only knows the basic addition facts but can also devise explanations of answers that typically draw on additive structure such as five-structure, ten-structure and doubles structure of even numbers. In the case of fractions, **grounded habituation** might involve habituated knowledge such as $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, along with being able to explain that $\frac{3}{4}$ is the answer because $\frac{1}{2}$ and $\frac{2}{4}$ are commensurate fractions.

Notating

In some instructional situations involving fractions, teachers ask their students to write or draw in order to symbolize their reasoning or to show a problem-solving strategy. This can lead to students developing an informal notation that they incorporate into their learning about fractions. We refer to this student activity as **notating**. **Notating** in this way can support advancements in students' conceptual knowledge of fractions. For example, a student might draw an arrow between a rectangle representing $\frac{1}{5}$ and a rectangle representing the whole, and then might label the rectangle '5 times' to notate the size relationship between $\frac{1}{5}$ and the whole. **Notating** that both students and teachers do should be a trace of their reasoning with pictures and mental images (representations of quantities), and then eventually come to stand in for reasoning.

Refining Computational Strategies

In the case of whole number arithmetic, students can learn intuitive mental strategies for adding and subtracting in the range 0–100 which are relatively sophisticated. For example, a student solves 48 + 35 in the following three steps: 48 + 30 = 78; 78 + 2 = 80; 80 + 3 = 83 (referred to as a 'jump' strategy). At an earlier time the student might have added three 10s singly (48 + 10; 58 + 10; 68 + 10). The former strategy involves curtailment of the latter strategy. This exemplifies *refining a computational strategy*. In the case of arithmetic involving fractions there are many examples of refinement of a computational strategy. For example, solving $\frac{7}{8} + \frac{5}{6}$ might involve an elaborated procedure to determine that twenty-fourths (or forty-eighths) can serve as a common measurement unit for eighths and sixths. Mathematizing in this case could involve finding the common measurement unit in one step. More broadly, learning the standard algorithms for adding, subtracting, multiplying and dividing involving fractions can involve the progressive refinement of computational strategies.

Structuring Numbers

Structuring occurs when the learner mentally imposes some form of arithmetical structure on numbers. A young learner who can say the number words from 1 to beyond 12, and can count a small collection of objects, might regard 17 as consisting of 17 ones. Contrast this with the learner who regards 17 as $10 + 7$, $10 + 5 + 2$, $8 + 8 + 1$, or $20 - 2 - 1$. The latter learner has what is called additive structure. Similarly, the student who can regard 24 as 2×12, 4×6, $240 \div 10$ and so on, has developed multiplicative structure. Structuring in the case of whole numbers can be extended to additive and multiplicative structuring involving fractions; for example, regarding seven-fifths as both $\frac{7}{5}$ and $1\frac{2}{5}$.

Unitizing

Unitizing refers to the mental act of regarding a collection as a unit. The collection could be a strip of ten dots, or three sevenths of a piece of string. The dimension of **unitizing** and units coordination are referred to extensively in this book, as explained in Chapters 2 and 3.